

Convergence of Brownian Motions on Metric Measure Spaces Under Riemannian Curvature–Dimension Conditions

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Abstract

We show that the pointed measured Gromov convergence of the underlying spaces implies (or under some condition, is equivalent to) the weak convergence of Brownian motions under Riemannian Curvature-Dimension (RCD) conditions. This paper is an improved and jointed version of the previous two manuscripts [71] and [72]. The improvements extend our results to the case of σ -finite reference measures and to the case that initial distributions of Brownian motions are possible to be dirac measures.

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1 Introduction

1.1 Motivation

The aim of this paper is to characterize a probabilistic convergence of diffusion processes in terms of a geometric convergence of the underlying spaces. As main results, we show that the measured Gromov convergence of the underlying spaces implies (or under some condition, is equivalent to) the weak convergence of Brownian motions under Riemannian Curvature-Dimension (RCD) conditions.

The RCD conditions are generalization of lower Ricci curvature bounds and admits various finite- and infinite-dimensional non-smooth spaces. These spaces typically appear as Gromov-Hausdorff (GH) limit spaces of Riemannian manifolds whose Ricci curvatures are uniformly bounded from below. Such spaces may have singularities and are generally no more manifolds (in fact, we admit dense singularities in Example 4.3). Furthermore the RCD framework includes infinite-dimensional spaces such as Hilbert spaces with log-concave measures (related to various stochastic partial differential equations) (see further in Section 4).

A natural question in probability theory is whether one can construct a Brownian motion on these non-smooth spaces, and if one can construct it, what properties it has. By recent developments of analysis on metric measure spaces, we can construct Brownian motions on RCD spaces by using a certain quadratic form, what is called *Cheeger energy*. This is a generalization of Dirichlet energy on smooth manifolds and induces a quasi-regular strongly local conservative symmetric Dirichlet form (Ambrosio-Gigli-Savaré [6, 7, 4]), which is determined only by the underlying metric measure structure.

One of the important problems for Brownian motions on these non-smooth spaces is to characterize the weak convergence of Brownian motions in terms of some geometrical convergence of the underlying spaces, which we call the *stability* of Brownian motions. One motivating example is: let a sequence of Riemannian manifolds (with lower Ricci curvature bounds) converge to a (possibly non-smooth) metric measure space in the Gromov-Hausdorff (GH) sense. If so, the question is whether a sequence of Brownian motions also converge weakly to the Brownian motion (if it exists) on the GH-limit space, or not.

The significance of the stability can be explained from several different perspectives. From the standpoint of limit theorems of stochastic processes, the stability is interpreted as

a geometric characterization of invariance principles for Brownian motions in the sense that Brownian motions on limit spaces are approximated by Brownian motions on converging spaces. From the viewpoint of “well-definedness”, the stability also enables us to verify that Brownian motions in limit spaces are “well-defined” in the sense that Brownian motions intrinsically defined by Cheeger energies on limit spaces coincide with limit processes of Brownian motions on approximating spaces. From the perspective that Brownian motions are considered as “a map” assigning laws of diffusions (i.e., probability measures on path spaces) to each metric measure space, the stability reveals the interesting fact that this map is continuous with respect to the corresponding topologies (e.g., GH-topology of metric measure spaces/weak topology of probability measures on path spaces), which is one ideal aspect of Brownian motions but has not been focused on so much until now.

The main contribution of this paper is to prove the stability of Brownian motions in the general framework of RCD spaces, whereby many singular spaces are included. Moreover, we show several equivalences of the weak convergence of Brownian motions and the pmG convergence of the underlying spaces. For references to other investigations regarding the stability problem, see the historical remarks (Section 1.3 below).

1.2 Main Results

In this paper, we always consider *pointed metric measure (p.m.m.) spaces* $\mathcal{X} = (X, d, m, \bar{x})$ whereby

(X, d) is a complete separable geodesic metric space with nonnegative and nonzero Borel measure m which is finite on all bounded sets, and \bar{x} is a fixed point in $\text{supp}[m]$. (1.1)

Our main results consist of two parts, one is for $\text{RCD}(K, \infty)$ spaces, and the other is for $\text{RCD}^*(K, N)$ spaces. The latter condition is stronger than the former one. We first state the results for $\text{RCD}(K, \infty)$ spaces. For the main theorems, we assume the following condition:

Assumption 1.1 Let $K \in \mathbb{R}$ and $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. Let $\{\mathcal{X}_n\}_{n \in \bar{\mathbb{N}}} = \{(X_n, d_n, m_n, \bar{x}_n)\}_{n \in \bar{\mathbb{N}}}$ be a sequence of p.m.m. spaces satisfying (1.1) and $\text{RCD}(K, \infty)$ condition.

The notion of $\text{CD}(K, \infty)$ spaces was introduced by Sturm [68] and Lott–Villani [48], and the notion of $\text{RCD}(K, \infty)$ spaces was introduced by Ambrosio–Gigli–Savaré [7] and Ambrosio–Gigli–Mondino–Rajala [4]. The $\text{CD}(K, \infty)$ condition is a generalization of “ $\text{Ric} \geq K$ ” for metric measure spaces in terms of the K -convexity of the entropy on the Wasserstein spaces. Furthermore $\text{RCD}(K, \infty)$ condition means the $\text{CD}(K, \infty)$ and that the Cheeger energy is *quadratic*. We will explain the precise definition in Subsection 2.4. RCD spaces admits the GH limit spaces of Riemannian manifolds with lower Ricci curvature bounds, and also admit Alexandrov spaces (metric spaces satisfying a generalized notion of “sectional curvature $\geq K$ ”) (Petrinin [56] and Zhang–Zhu [74]). Moreover, not only finite-dimensional spaces, but also several infinite-dimensional spaces related to stochastic partial differential equations are included such as Hilbert spaces with log-concave measures (Ambrosio–Savaré–Zambotti [10]).

Under Assumption 1.1, we can always take constants $c_1, c_2 > 0$ dependent only on K satisfying the following volume growth estimate (see [68, Theorem 4.24])

$$m_n(B_r(\bar{x}_n)) \leq c_1 e^{c_2 r^2}, \quad \forall r > 0. \quad (1.2)$$

Here we mean $B_r(\bar{x}_n) := \{x \in X : d(x, \bar{x}_n) < r\}$. Taking $C > c_2$, we set a *weighted measure* \tilde{m}_n as follows:

$$z_n := \int_{X_n} e^{-Cd_n^2(x, \bar{x}_n)} dm_n(x), \quad \text{and} \quad \tilde{m}_n := \begin{cases} \frac{1}{z_n} e^{-Cd_n^2(\cdot, \bar{x}_n)} m_n, & \text{if } m_n(X_n) = \infty, \\ \frac{1}{m_n(X_n)} m_n, & \text{if } m_n(X_n) < \infty. \end{cases} \quad (1.3)$$

Under Assumption 1.1, the Cheeger energy Ch_n on $\mathcal{X}_n = (X_n, d_n, m_n, \bar{x}_n)$ (see Subsection 2.4.2) induces a quasi-regular conservative symmetric strongly local Dirichlet form, and there exists a conservative diffusion process $(\{\mathbb{P}_n^x\}_{x \in X_n}, \{B_t^n\}_{t \geq 0})$ on \mathcal{X}_n , which is unique at quasi-every starting point in X_n (see [7] for the case of $m(X) = 1$ and see [4] for the σ -finite case). We call $(\{\mathbb{P}_n^x\}_{x \in X_n}, \{B_t^n\}_{t \geq 0})$ *Brownian motion* on \mathcal{X}_n .

The main theorem states that the weak convergence of the Brownian motions can be characterized by the pmG convergence of the underlying spaces under Assumption 1.1 (we will give the definition of the pmG convergence in Subsection 2.3).

Theorem 1.2 *Suppose that Assumption 1.1 holds. Then the following (i) and (ii) are equivalent:*

(i) (pmG Convergence of the Underlying Spaces)

The p.m.m. spaces \mathcal{X}_n converges to $\mathcal{X}_\infty = (X_\infty, d_\infty, m_\infty, \bar{x}_\infty)$ in the pmG sense.

(ii) (Weak Convergence of the Laws of Brownian Motions)

There exist a complete separable metric space (X, d) and isometric embeddings $\iota_n : X_n \rightarrow X$ ($n \in \bar{\mathbb{N}}$) so that $\iota_n(\bar{x}_n) \rightarrow \iota_\infty(\bar{x}_\infty)$, and

$$(\iota_n(B^n), \mathbb{P}_n^{\tilde{m}_n}) \rightarrow (\iota_\infty(B^\infty), \mathbb{P}_\infty^{\tilde{m}_\infty}) \quad \text{weakly in } \mathcal{P}(C([0, \infty); X)). \quad (1.4)$$

Here $(\iota_n(B^n), \mathbb{P}_n^{\tilde{m}_n})$ means the law of the embedded Brownian motion $\iota_n(B^n)$ with the initial distribution \tilde{m}_n and $\mathcal{P}(C([0, \infty); X))$ denotes the set of all Borel probability measures on the continuous path space $C([0, \infty); X)$.

Remark 1.3 Several remarks for Theorem 1.2 are given below.

- (i) The $\text{RCD}(K, \infty)$ condition is stable under the pmG convergence (see [32, Theorem 7.2]), and therefore the limit space \mathcal{X}_∞ also satisfies the $\text{RCD}(K, \infty)$ condition so that the Brownian motion can be defined also on the limit space \mathcal{X}_∞ .
- (ii) The pmG convergence is weaker than the measured Gromov-Hausdorff convergence. See [32, Proposition 3.33].

In the statement (ii) in Theorem 1.2, the initial distribution is absolutely continuous with respect to the reference measure m_n . It is natural in the next step to ask how the case of the dirac measure $\delta_{\bar{x}_n}$ is, which means the Brownian motions start at the point \bar{x}_n . We introduce several conditions below:

(A) For any $n \in \mathbb{N}$, $m_n(X_n) < \infty$.

(B) For any $r > 0$,

$$\sup_{n \in \mathbb{N}} \|p_n(t, \bar{x}_n, \cdot)\|_{\infty, B_r(\bar{x}_n)} < \infty,$$

whereby $p_n(t, x, y)$ is the density of the transition probability $p_n(t, x, dy)$ with respect to the reference measure m_n , and $\|\cdot\|_{\infty, B_r(\bar{x}_n)}$ means the essential supremum on the ball $B_r(\bar{x}_n)$.

Now we state the second main result.

Theorem 1.4 *Suppose that Assumption 1.1 holds. If, moreover, either (A), or (B) holds, then (i) (thus also (ii)) in Theorem 1.2 implies the following (iii)_{>0}:*

(iii)_{>0} **(Weak Convergence of the Laws of Brownian Motions starting at points)**

There exist a complete separable metric space (X, d) and isometric embeddings $\iota_n : X_n \rightarrow X$ ($n \in \mathbb{N}$) so that it holds that

$$(\iota_n(B^n), \mathbb{P}_n^{\bar{x}_n}) \rightarrow (\iota_\infty(B^\infty), \mathbb{P}_\infty^{\bar{x}_\infty}) \quad \text{weakly in } \mathcal{P}(C((0, \infty); X)). \quad (1.5)$$

Remark 1.5 Several remarks for Theorem 1.4 are given below.

- (i) Note that, in (1.5), the time interval of the path space is not $[0, \infty)$ but $(0, \infty)$. This is due to the ambiguity in the starting points of Markov processes associated with Dirichlet forms. However, since Brownian motions on $\text{RCD}(K, \infty)$ spaces are conservative and the heat kernel $p_n(\varepsilon, x, dy)$ is absolutely continuous with respect to m_n for every x and every $\varepsilon > 0$, the laws $(\iota_n(B^n), \mathbb{P}_n^x)$ live on $C((0, \infty); X)$ for *every* $x \in X_n$ (not only *quasi-every* x) (see Section 3). Therefore the statement (iii)_{>0} in Theorem 1.4 makes sense without quasi-every ambiguity of starting points \bar{x}_n if we restrict the time interval $[0, \infty)$ to $(0, \infty)$.
- (ii) The condition (B) is satisfied for any $\text{RCD}^*(K, N)$ spaces because of local Gaussian heat kernel estimates, which follow from the local volume doubling property and the local Poincaré inequality according to Sturm [66] (see also Jiang–Li–Zhang [39]).
- (iii) If the following uniform ultra-contractivity of the heat semigroup $\{H_t\}_{t \geq 0}$ holds, then the condition (B) holds (see [6, Proposition 6.4]): there exists a $p > 1$ so that, with some positive constant $C(t, K)$ dependent only on t and K , we have

$$\|H_t f\|_p \leq C(t, K) \|f\|_1, \quad \forall f \in L^1(X, m), \quad \forall t > 0.$$

We have examples satisfying the ultra-contractivity which is a $\text{RCD}(K, \infty)$ space but not a $\text{RCD}^*(K, N)$ space for any $1 < N < \infty$. Let $\mathcal{X}_\alpha = (\mathbb{R}, |\cdot - \cdot|, C_\alpha \exp\{-|\cdot|^\alpha\})$ whereby $\alpha \in \{2, 4, 6, \dots\}$ is an even number and C_α is the normalizing constant. For any $\alpha > 2$, it is known that \mathcal{X}_α satisfies the ultra-contractivity of the heat semigroup (Bakry–Bolley–Gentil–Maheux [15]) and satisfies the $\text{RCD}(0, \infty)$ condition, but not $\text{RCD}^*(K, N)$ for any finite $1 < N < \infty$.

By Theorem 1.4 and Remark 1.5, we have the following corollary.

Corollary 1.6 *If a sequence of p.m.m. spaces \mathcal{X}_n satisfies (1.1) and $\text{RCD}^*(K, N)$ with $1 < N < \infty$, then (i) (thus also (ii)) in Theorem 1.2 implies (iii)_{>0} in Theorem 1.4.*

Next we consider the converse implication that the weak convergence of Brownian motions induces the pmG convergence of the underlying spaces. Let $p_n(t, x, y)$ be the heat kernel for the p.m.m. space $\mathcal{X}_n = (X_n, d_n, m_n, \bar{x}_n)$. Let us consider the following condition: there exists $t_* > 0$ and a constant M so that

$$\sup_{n \in \mathbb{N}} p_n(t_*, \bar{x}_n, \bar{x}_n) < M < \infty. \quad (1.6)$$

Note that, since $p_n(t, x, x)$ is non-increasing function in t , if we find the time t_* satisfying (1.6), then for any $t > t_*$, the estimate (1.6) holds. For instance, all $\text{RCD}^*(K, N)$ spaces with $1 < N < \infty$ satisfies (1.6) because of the local Gaussian heat kernel estimate. Let $\text{diam}(X_n)$ denote the diameter of X_n : $\text{diam}(X_n) := \inf\{A \geq 0 : \sup_{x, y \in X_n} d_n(x, y) \leq A\}$. We now state the following theorem:

Theorem 1.7 *Suppose that Assumption 1.1 and the condition (1.6) hold. Let $0 < D < \infty$. If, moreover, either $K > 0$, or $\sup_{n \in \mathbb{N}} \text{diam}(X_n) < D$ holds, then (iii) $_{>0}$ in Theorem 1.4 implies (i) and (ii) in Theorem 1.2 (therefore all the statements (i), (ii) and (iii) $_{>0}$ are equivalent).*

Now we state the result for $\text{RCD}^*(K, N)$ spaces. We assume the following condition:

Assumption 1.8 Let $K \in \mathbb{R}$, $N > 1$, and $0 < D < \infty$. Let $\{\mathcal{X}_n\}_{n \in \mathbb{N}} = \{(X_n, d_n, m_n, \bar{x}_n)\}_{n \in \mathbb{N}}$ be a sequence of p.m.m. spaces satisfying (1.1) and $\text{RCD}^*(K, N)$ condition with $\sup_{n \in \mathbb{N}} \text{diam}(X_n) < D$.

The $\text{RCD}^*(K, N)$ (*Riemannian curvature-dimension condition*) space was first introduced by Erbar–Kuwada–Sturm [26] and Ambrosio–Mondino–Savaré [9] which is the class of metric measure spaces satisfying the *reduced curvature-dimension condition* $\text{CD}^*(K, N)$ with the Cheeger energies being *quadratic*. Roughly speaking, $\text{RCD}^*(K, N)$ condition is a generalization of “Ricci $\geq K$ ” and “dim $\leq N$ ” to metric measure spaces. Various examples are known to be included in $\text{RCD}^*(K, N)$ such as the mGH limit spaces of N -dimensional Riemannian manifolds with $\text{Ric} \geq K$, N -dimensional Alexandrov spaces with $\text{Curv} \geq K$, cone spaces and warped product spaces (Ketterer [43, 44]).

Under Assumption 1.8, the Brownian motion becomes a Feller process. Therefore Brownian motions can be unique at every starting point. We now state the following theorem.

Theorem 1.9 *Suppose that Assumption 1.8 holds. Then all the four statements of (i), (ii) in Theorem 1.2, (iii) $_{>0}$ in Theorem 1.4 and the following (iii) $_{\geq 0}$ are equivalent:*

(iii) $_{\geq 0}$ (**Weak Convergence of the Laws of Brownian Motions starting at points**)

There exist a compact metric space (X, d) and isometric embeddings $\iota_n : X_n \rightarrow X$ ($n \in \mathbb{N}$) so that

$$(\iota_n(B^n), \mathbb{P}_n^{\bar{x}_n}) \rightarrow (\iota_\infty(B^\infty), \mathbb{P}_\infty^{\bar{x}_\infty}) \quad \text{weakly in } \mathcal{P}(C([0, \infty); X)). \quad (1.7)$$

Remark 1.10 We give several remarks for Theorem 1.9.

- (i) The $\text{RCD}^*(K, N)$ condition is stable under the pmG convergence (see [26]), and therefore the limit space \mathcal{X}_∞ also satisfies the $\text{RCD}^*(K, N)$ condition so that the Brownian motion can be defined at every starting point also on the limit space \mathcal{X}_∞ .
- (ii) The pmG convergence (see Definition 2.1) is equivalent to the pointed measured Gromov-Hausdorff convergence under Assumption 1.8 (see [32, Proposition 3.33]).

1.3 Historical Remarks

Remark 1.11 Several historical remarks are given below.

- (i) In Ambrosio–Savaré–Zambotti [10, Theorem 1.5], under the weak convergence of the underlying reference measures, they investigated the weak convergence of Brownian motions on Hilbert spaces with log-concave measures, which is a specific case of $\text{RCD}(0, \infty)$ spaces. Our results (Theorem 1.2 and 1.4) generalize their result [10, Theorem 1.5] for general $\text{RCD}(K, \infty)$ spaces.
- (ii) In Ogura [53], under the condition of uniform upper bounds for heat kernels (not necessarily lower bound of Ricci curvatures) and the Kasue–Kumura (KK) spectral convergence, he studied the weak convergence of the laws of *time-discretized* Brownian motions on weighted compact Riemannian manifolds. The KK spectral convergence roughly means a uniform convergence of heat kernels and stronger than the mGH convergence. In his case, the Ricci curvature is not necessarily bounded from below and the limit process may be a jump process ([53, 4.6]). The time-discretization is one possible approach for a convergence of stochastic processes on varying spaces, while we adopt in this paper a different approach, i.e., embedding into one common metric space X .
- (iii) If we do not assume RCD conditions for a sequence of the underlying metric measure spaces, then limit processes are not necessarily diffusions. In Ogura–Tomisaki–Tuchiya [54], they considered a sequence of Euclidean spaces $(\mathbb{R}^d, \|\cdot\|_2)$ with certain underlying measures μ_n whereby $\{(\mathbb{R}^d, \|\cdot\|_2, \mu_n)\}_{n \in \mathbb{N}}$ does not necessarily satisfy RCD conditions. They showed that diffusion processes on \mathbb{R}^d associated with the corresponding local Dirichlet forms converge to jump processes (or generally jump-diffusion processes) corresponding to certain non-local Dirichlet forms.
- (iv) In Albeverio–Kusuoka [2], diffusion processes associated with SDEs on thin tubes in \mathbb{R}^d were studied. When thin tubes shrink to a spider graph, diffusion processes converge weakly to a one-dimensional diffusion on this spider graph. Their setting does not satisfy the RCD condition since spider graphs branch at points of conjunctions but RCD spaces are essentially non-branching (see [61, Theorem 1]).
- (v) In Athreya–Löhr–Winter [13], the weak convergence of certain Markov processes on tree-like spaces was studied. When tree-like spaces converge in *Gromov-vague* sense, the corresponding processes also converge weakly. Their tree-like spaces admit 0-hyperbolic spaces, which are not necessarily included in RCD spaces.
- (vi) In Suzuki [71], the author investigated the weak convergence of continuous stochastic processes on metric spaces converging in the Lipschitz distance. The Lipschitz convergence is stronger than the measured Gromov convergence (see [34, Section 3.C]).

Finally we list related studies not mentioned in Remark 1.11. In Stroock–Varadhan [63], Stroock–Zheng [64] and Burdzy–Chen [21], approximations of diffusion processes on \mathbb{R}^d by discrete Markov chains on $(1/n)\mathbb{Z}^d$ were investigated. In Bass–Kumagai–Uemura [17] and Chen–Kim–Kumagai [23], they studied approximations of jump processes on proper metric spaces by Markov chains on discrete graphs. Approximations of Markov processes on ultra-metric spaces were explored in Suzuki [69]. In Pinsky [58], he studied approximations of Brownian motions on Riemannian manifolds by random walks, while the case of sub-Riemannian manifolds was investigated by Gordina and Laetsch [33]. In Croydon–Hambly–Kumagai [24], in which it was assumed that a sequence of resistance forms converges with respect to the GH-vague topology and satisfies a uniform volume doubling condition, they showed the weak convergence of corresponding Brownian motions and local times. There are

many studies about scaling limits of random processes on random environments (see, e.g., Kumagai [45] and references therein).

1.4 Organization of the Paper

The paper is structured as follows: First, the notation is fixed and preliminary facts are recalled in Section 2 (no new results are included). Each of the subsections define important aspects of the research in this paper, namely: basic notations and basic definitions (Subsection 2.1); L^2 -Wasserstein distance (Subsection 2.2); pmG convergence (Subsection 2.3); $\text{RCD}(K, \infty)$ and $\text{RCD}^*(K, N)$ spaces (Subsection 2.4); L^2 -convergence of the heat semigroup (Subsection 2.5). In Section 3, we state several properties about Brownian motions on RCD spaces. In Section 4 we present examples in which Assumptions 1.1 and 1.8 are satisfied. These examples consist of weighted Riemannian manifolds and its pmG limit spaces, Alexandrov spaces, and Hilbert spaces with log-concave probability measures. In Section 5, we give the proof of Theorem 1.2. In Section 6, we show the proof of Theorem 1.4. In Section 7, we prove Theorem 1.7. Finally, in Section 8, we prove Theorem 1.9.

2 Notation & Preliminary Results

2.1 Notation

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ be the set of natural numbers and the set of extended natural numbers, respectively. Let (X, d) be a complete separable metric space. We write $B_r(x) = \{y \in X : d(x, y) < r\}$ for an open ball centered at $x \in X$ with radius $r > 0$. By using $\mathcal{B}(X)$, we denote the family of all Borel sets in (X, d) ; and by $\mathcal{B}_b(X)$, the set of real-valued bounded Borel-measurable functions on X . Let $C(X)$ denote the set of real-valued continuous functions on X , while $C_b(X)$, $C_0(X)$ and $C_{bs}(X)$ denote the subsets of $C(X)$ consisting of bounded functions, functions with compact support, and bounded functions with bounded support, respectively. The set $\mathcal{P}(X)$ denotes all Borel probability measures on X . The set of continuous functions on $[0, \infty)$ valued in X is denoted by $C([0, \infty), X)$.

A continuous curve $\gamma : [a, b] \rightarrow X$ is said to be *connecting x and y* if $\gamma_a = x$ and $\gamma_b = y$. A continuous curve $\gamma : [a, b] \rightarrow X$ is said to be a *minimal geodesic* if

$$d(\gamma_t, \gamma_s) = \frac{|s - t|}{|b - a|} d(\gamma_a, \gamma_b) \quad a \leq t \leq s \leq b.$$

In particular, if $\frac{d(\gamma_a, \gamma_b)}{|b - a|}$ can be replaced by 1, we say that γ is *unit-speed*.

Let $\text{supp}[m] = \{x \in X : m(B_r(x)) > 0, \forall r > 0\}$ denote the support of m . Let (Y, d_Y) be a complete separable metric space. For a Borel measurable map $f : X \rightarrow Y$, let $f_{\#}m$ denote the push-forward measure on Y :

$$f_{\#}m(B) = m(f^{-1}(B)) \quad \text{for any Borel set } B \in \mathcal{B}(Y).$$

2.2 L^p -Wasserstein Space

Let (X_i, d_i) ($i = 1, 2$) be complete separable metric spaces and $1 < p < \infty$. For $\mu_i \in \mathcal{P}(X_i)$, a probability measure $q \in \mathcal{P}(X_1 \times X_2)$ is called a *coupling of μ_1 and μ_2* if

$$\pi_{1\#}q = \mu_1 \quad \text{and} \quad \pi_{2\#}q = \mu_2,$$

whereby π_i ($i = 1, 2$) is the projection $\pi_i : X_1 \times X_2 \rightarrow X_i$ as $(x_1, x_2) \mapsto x_i$. By using $\Pi(\mu, \nu)$, we denote the set of all coupling of μ and ν .

Let (X, d) be a complete separable metric space. Let $\mathcal{P}_p(X)$ be the subset of $\mathcal{P}(X)$ consisting of all Borel probability measures μ on X with finite second moment:

$$\int_X d^p(x, \bar{x}) d\mu(x) < \infty \quad \text{for some (and thus any) } \bar{x} \in X.$$

We endow $\mathcal{P}_p(X)$ with the quadratic transportation distance W_p , called *L^p -Wasserstein distance*, defined as follows:

$$W_p(\mu, \nu) = \left(\inf_{q \in \Pi(\mu, \nu)} \int_{X \times X} d^p(x, y) dq(x, y) \right)^{1/p}. \quad (2.1)$$

A coupling $q \in \Pi(\mu, \nu)$ is called *an optimal coupling* if q attains the infimum in the equality (2.1). It is known that, for any μ, ν , there always exists an optimal coupling q of μ and ν (e.g., [73, §4]). It is known that $(\mathcal{P}_p(X), W_p)$ is a complete separable metric space for $1 < p < \infty$ (e.g., [73, Theorem 6.18]).

2.3 Pointed Measured Gromov Convergence

We recall the definition of pmG convergence introduced in Gigli-Mondino-Savaré [32].

Definition 2.1 ([32]) (pmG Convergence) A sequence of p.m.m. spaces $\mathcal{X}_n = (X_n, d_n, m_n, \bar{x}_n)$ satisfying (1.1) is said to be *convergent to $\mathcal{X}_\infty = (X_\infty, d_\infty, m_\infty, \bar{x}_\infty)$ in the pointed measured Gromov (pmG) sense* if there exist a complete separable metric space (X, d) and isometric embeddings $\iota_n : X_n \rightarrow X$ ($n \in \bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$) satisfying

$$\iota_n(\bar{x}_n) \rightarrow \iota_\infty(\bar{x}_\infty) \in \text{supp}[m_\infty], \quad \text{and} \quad \int_X f d(\iota_n \# m_n) \rightarrow \int_X f d(\iota_\infty \# m_\infty), \quad (2.2)$$

for any bounded continuous function $f : X \rightarrow \mathbb{R}$ with bounded support. The subscript $\#$ denotes the operation of the push-forward of measures.

Remark 2.2 We give several remarks for Definition 2.1.

- (i) The pmG convergence is weaker than the pointed measured Gromov-Hausdorff (pmGH) convergence ([32, Proposition 3.30, Example 3.31]). If $\text{supp}[m_\infty] = X_\infty$ and $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$ satisfies a uniform doubling condition, then pmG and pmGH coincide [32, Proposition 3.33].
- (ii) The pmG convergence can be metrizable by a distance $p\mathbb{G}_W$ on the collection \mathbb{X} of all isomorphism classes of p.m.m. spaces ([32, Definition 3.13]). The space $(\mathbb{X}, p\mathbb{G}_W)$ becomes a complete and separable metric space ([32, Theorem 3.17]).

2.4 RCD Spaces

In this subsection, the definition of the $\text{RCD}(K, \infty)$ condition is recalled by following [32]. Several properties satisfied on $\text{RCD}(K, \infty)$ spaces are also recalled. In this section, we always assume that $\mathcal{X} = (X, d, m, \bar{x})$ be a p.m.m. space satisfying (1.1) and (1.2).

2.4.1 Relative Entropy

The relative entropy functional $\text{Ent}_m : \mathcal{P}_2(X) \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is defined as follows:

$$\text{Ent}_m(\mu) = \begin{cases} \int_X \frac{d\mu}{dm} \log\left(\frac{d\mu}{dm}\right) dm & \text{if } \mu \ll m, \\ +\infty & \text{otherwise,} \end{cases}$$

whereby $d\mu/dm$ denotes the Radon–Nikodym derivative. We write $D(\text{Ent}_m) := \{\mu \in \mathcal{P}_2(X) : \text{Ent}_m(\mu) < \infty\}$. Although m is not a probability measure, the entropy Ent_m is well-posed and lower semicontinuous because of the condition (1.2). In fact, recall (1.3):

$$z := \int_X e^{-C d^2(x, \bar{x})} dm(x), \quad \text{so that} \quad \tilde{m} := \frac{1}{z} e^{-C d^2(\cdot, \bar{x})} m. \quad (2.3)$$

Then we can check that, for any $\rho m = \mu \in D(\text{Ent}_m)$ with $\rho = \frac{d\mu}{dm}$, it holds that $\mu = z \rho e^{C d^2(\cdot, \bar{x})} \tilde{m}$. Then we have

$$\text{Ent}_m(\mu) = \text{Ent}_{\tilde{m}}(\mu) - C \int_X d^2(\cdot, \bar{x}) d\mu - \log z, \quad (2.4)$$

which implies that Ent_m is well-posed and lower semicontinuous with respect to W_2 -topology. See [32, §4.1.1] for more details.

2.4.2 Cheeger Energy

Let us recall the Cheeger energy Ch on (X, d, m, \bar{x}) , which is defined as the limit of the integral of local Lipschitz constants. Let $\text{Lip}(X)$ denote the set of real-valued Lipschitz continuous functions on X . For $f \in \text{Lip}(X)$, the local Lipschitz constant $|\nabla f| : X \rightarrow \mathbb{R}$ is defined as follows:

$$|\nabla f|(x) = \begin{cases} \limsup_{y \rightarrow x} \frac{|f(y) - f(x)|}{d(y, x)} & \text{if } x \text{ is not isolated,} \\ 0 & \text{otherwise.} \end{cases}$$

Then *Cheeger energy* is defined as follows: (see [6, 4])

$$\begin{aligned} \text{Ch}(f) &= \frac{1}{2} \inf \left\{ \liminf_{n \rightarrow \infty} \int |\nabla f_n|^2 dm : f_n \in \text{Lip}(X), \int_X |f_n - f|^2 dm \rightarrow 0 \right\} \\ W^{1,2}(X, d, m) &= \{f \in L^2(X : m) : \text{Ch}(f) < \infty\}. \end{aligned}$$

We note that if $\text{Ch}(f) < \infty$, then the Cheeger energy can be written as an integral form by using what is called *minimal weak upper gradient* $|\nabla f|_w$ (see [6, 4]):

$$\text{Ch}(f) = \frac{1}{2} \int_X |\nabla f|_w^2 dm, \quad \forall f \in W^{1,2}(X, d, m). \quad (2.5)$$

Remark 2.3 Note that $\text{Ch} : L^2(X, m) \rightarrow [0, +\infty]$ is a lower semi-continuous and convex functional, but not necessarily a quadratic form. This means that $(W^{1,2}(X, d, m), \sqrt{2\text{Ch}}(\cdot))$ is a Banach space, but not necessarily a Hilbert space.

2.4.3 RCD(K, ∞) Spaces

Definition 2.4 ($\text{CD}(K, \infty)$ and $\text{RCD}(K, \infty)$)

(i) [Sturm [68], Lott–Villani [48]]

We say that (X, d, m) satisfies *the curvature-dimension condition* $\text{CD}(K, \infty)$ for $K \in \mathbb{R}$ if, for each $\mu_0, \mu_1 \in D(\text{Ent}_m)$, there exists a W_2 -geodesic $\{\mu_t\}_{t \in [0,1]} \subset D(\text{Ent}_m)$ connecting μ_0 and μ_1 so that

$$\text{Ent}_m(\mu_t) \leq (1-t)\text{Ent}_m(\mu_0) + t\text{Ent}_m(\mu_1) - \frac{K}{2}t(1-t)W_2^2(\mu_0, \mu_1). \quad (2.6)$$

(ii) [Ambrosio–Gigli–Savaré [7, Theorem 5.1] & Ambrosio–Gigli–Mondino–Rajala [4, Theorem 6.1]]

We say that (X, d, m) satisfies *the Riemannian curvature-dimension condition* $\text{RCD}(K, \infty)$ if the following two conditions hold:

(a) $\text{CD}(K, \infty)$

(b) the infinitesimal Hilbertianity, that is, the Cheeger energy Ch is a quadratic form:

$$2\text{Ch}(u) + 2\text{Ch}(v) = \text{Ch}(u+v) + \text{Ch}(u-v). \quad (2.7)$$

for any $u, v \in W^{1,2}(X, d, m)$.

Both of $\text{CD}(K, \infty)$ and $\text{RCD}(K, \infty)$ conditions are stable under the pmG convergence.

Theorem 2.5 ([68, 7, 4]) (**Stability of the $\text{RCD}(K, \infty)$ condition**)

Let \mathcal{X}_n be a sequence of p.m.m. spaces satisfying the $\text{RCD}(K, \infty)$ condition. If \mathcal{X}_n converges to \mathcal{X}_∞ in the pmG sense, then the limit space \mathcal{X}_∞ also satisfies the $\text{RCD}(K, \infty)$ condition.

2.4.4 W_2 -gradient Flow of Relative Entropy

In this subsection, we recall a flow on the L^2 -Wasserstein space $(\mathcal{P}_2(X), W_2)$, which is called *heat flow* and constructed by the gradient flow of the relative entropy functional. We also recall the stability of the heat flows under the pmG convergence. We follow [6, 4] in this section.

The descendent slope $|D^-\text{Ent}_m| : \mathcal{P}_2(X) \rightarrow [-\infty, \infty]$ of the relative entropy Ent_m is defined as follows:

$$|D^-\text{Ent}_m|(\mu) = \begin{cases} \limsup_{W_2(\nu, \mu) \rightarrow 0} \frac{(\text{Ent}_m(\mu) - \text{Ent}_m(\nu))^+}{W_2(\nu, \mu)}, & \text{if } \mu \in D(\text{Ent}_m), \\ 0 & \text{if } \mu \text{ is isolated in } \mathcal{P}_2(X) \\ +\infty & \text{if } \mu \in \mathcal{P}_2(X) \setminus D(\text{Ent}_m), \end{cases}$$

whereby $(\cdot)^+$ means the positive part.

Let $\mathcal{X} = (X, d, m, \bar{x})$ be a $\text{CD}(K, \infty)$ space and $\bar{\mu} \in D(\text{Ent}_m)$. A curve $\mu : [0, \infty) \rightarrow D(\text{Ent}_m) \subset \mathcal{P}_2(X)$ is *the W_2 -gradient flow of Ent_m starting at $\bar{\mu}$* if μ is locally absolutely continuous in $(\mathcal{P}_2(X), W_2)$ with $\mu_0 = \bar{\mu}$ and

$$\text{Ent}_m(\mu_t) = \text{Ent}_m(\mu_s) + \frac{1}{2} \int_t^s |\dot{\mu}_r|^2 dr + \frac{1}{2} \int_t^s |D^-\text{Ent}_m|^2(\mu_r) dr, \quad 0 < \forall t < \forall s.$$

Under the $\text{CD}(K, \infty)$ condition, it is known that the gradient flow $\mu_t = \mathcal{H}_t \bar{\mu}$ of the relative entropy exists uniquely for any initial measure $\bar{\mu} \in D(\text{Ent}_m)$ and for any $t \geq 0$ ([6, 4]). We call $\{\mathcal{H}_t\}_{t \geq 0}$ *heat flow on $\mathcal{P}_2(X)$* .

Theorem 2.6 (Theorem 7.7 in [32]) (Stability of heat flows)

Let $\mathcal{X}_n = (X_n, d_n, m_n, \bar{x}_n)$ be a sequence of $\text{RCD}(K, \infty)$ spaces converging to $\mathcal{X}_\infty = (X_\infty, d_\infty, m_\infty, \bar{x}_\infty)$ in the pmG sense. If $\mu_n \in \mathcal{P}_2(\text{supp}[m_n]) \subset \mathcal{P}_2(X)$ converges to $\mu_\infty \in \mathcal{P}_2(\text{supp}[m_\infty]) \subset \mathcal{P}_2(X)$ in the W_2 -sense:

$$W_2(\iota_n \# \mu_n, \iota_\infty \# \mu_\infty) \rightarrow 0, \quad n \rightarrow \infty,$$

then the solution $\mu_t^n = \mathcal{H}_t^n(\mu_n)$ of the heat flow starting at μ_n converges to the limit one $\mu_t^\infty = \mathcal{H}_t^\infty(\mu_\infty)$ in the W_2 -sense:

$$W_2(\iota_n \# \mu_t^n, \iota_\infty \# \mu_t^\infty) \rightarrow 0, \quad n \rightarrow \infty, \quad \forall t \geq 0.$$

Here ι_n is an embedding $X_n \rightarrow X$ corresponding to the pmG convergence (see Definition 2.1).

2.4.5 L^2 -gradient Flow of Cheeger Energy

We now recall the L^2 -gradient flow of Cheeger energy by Hilbertian theory of gradient flows (see e.g., [8]). We also recall the important fact that the heat flow in the previous section and the L^2 -gradient flow of Cheeger energy in this section coincide under the $\text{CD}(K, \infty)$ condition.

For $f_0 \in L^2(X; m)$, there exists a locally Lipschitz map $t \mapsto f_t = H_t f_0 \in L^2(X; m)$ with $f_t \rightarrow f_0$ as $t \downarrow 0$ whose derivative satisfies

$$\frac{d}{dt} f_t \in -\partial^- \text{Ch}(f_t), \quad \text{a.e. } t > 0. \quad (2.8)$$

Here the subdifferential $\partial^- \text{Ch}$ of convex analysis is the multi-valued operator in $L^2(X; m)$ defined at all elements of the domain of the Cheeger energy $f \in W^{1,2}(X, d, m)$ by the family of inequalities

$$h \in \partial^- \text{Ch}(f) \iff \int_X h(g - f) dm \leq \text{Ch}(g) - \text{Ch}(f), \quad \forall g \in L^2(X; m).$$

The map $H_t : f_0 \mapsto f_t$ is uniquely determined by (2.8) and define a contraction semigroup (not necessarily linear) on $L^2(X; m)$. The flow $f_0 \mapsto f_t = H_t f$ is called *L^2 -gradient flow of the Cheeger energy*, and the semigroup $\{H_t\}_{t \geq 0}$ is called *heat semigroup*.

We recall that the L^2 -gradient flows of Cheeger energies and the W_2 -gradient flow of entropies are equivalent under the $\text{CD}(K, \infty)$ condition.

Theorem 2.7 [6, Theorem 9.3](see also [4]) *Let $\mathcal{X} = (X, d, m, \bar{x})$ be a p.m.m. space satisfying the $\text{CD}(K, \infty)$ condition. If $\mu_0 = f_0 m \in \mathcal{P}_2(X)$ with $f \in L^2(X; m)$, then*

$$\mathcal{H}_t(\mu_0) = (H_t f_0) m, \quad \forall t \geq 0.$$

2.4.6 $\text{RCD}^*(K, N)$ Spaces

In this subsection, the definition of the $\text{RCD}^*(K, N)$ condition and several properties satisfied by $\text{RCD}^*(K, N)$ spaces are recalled (see [9] and [26]).

For each $\theta \in [0, \infty)$, we set

$$\Theta_\kappa(\theta) = \begin{cases} \frac{\sin(\sqrt{\kappa}\theta)}{\sqrt{\kappa}} & \text{if } \kappa > 0, \\ \theta & \text{if } \kappa = 0, \\ \frac{\sinh(\sqrt{-\kappa}\theta)}{\sqrt{-\kappa}} & \text{if } \kappa < 0, \end{cases}$$

and set for $t \in [0, 1]$,

$$\sigma_\kappa^{(t)}(\theta) = \begin{cases} \frac{\Theta_\kappa(t\theta)}{\Theta_\kappa(\theta)} & \text{if } \kappa\theta^2 \neq 0 \text{ and } \kappa\theta^2 < \pi^2, \\ t & \text{if } \kappa\theta^2 = 0, \\ +\infty & \text{if } \kappa\theta^2 \geq \pi^2. \end{cases}$$

Let $P_\infty(X, d, m)$ be the subset of $\mathcal{P}_2(X)$ consisting of μ which is absolutely continuous with respect to m and has bounded support.

Definition 2.8 ([9, 14, 26]) ($\text{CD}^*(K, N)$ and $\text{RCD}^*(K, N)$)

- (i) We say that (X, d, m) satisfies *the reduced curvature-dimension condition* $\text{CD}^*(K, N)$ for $K, N \in \mathbb{R}$ with $N > 1$ if, for each pair $\mu_0 = \rho_0 m$ and $\mu_1 = \rho_1 m$ in $P_\infty(X, d, m)$, there exists an optimal coupling q of μ_0 and μ_1 and a geodesic $\mu_t = \rho_t m$ ($t \in [0, 1]$) in $(P_\infty(X, d, m), W_2)$ connecting μ_0 and μ_1 so that, for all $t \in [0, 1]$ and $N' \geq N$, we have

$$\begin{aligned} \int \rho_t^{-\frac{1}{N'}} d\mu_t &\geq \int_{X \times X} \left[\sigma_{K/N'}^{(1-t)}(d(x_0, x_1)) \rho_0^{-1/N'}(x_0) \right. \\ &\quad \left. + \sigma_{K/N'}^{(t)}(d(x_0, x_1)) \rho_1^{-1/N'}(x_1) \right] dq(x_0, x_1). \end{aligned} \quad (2.9)$$

- (ii) We say that (X, d, m) satisfies *the Riemannian curvature-dimension condition* $\text{RCD}^*(K, N)$ if the following two conditions hold:

(a) $\text{CD}^*(K, N)$

(b) the infinitesimal Hilbertianity, that is the Cheeger energy Ch is a quadratic form:

$$\begin{aligned} 2\text{Ch}(u) + 2\text{Ch}(v) &= \text{Ch}(u + v) + \text{Ch}(u - v), \\ \forall u, v &\in W^{1,2}(X, d, m). \end{aligned} \quad (2.10)$$

Remark 2.9 The $\text{RCD}^*(K, N)$ condition is stronger than the $\text{RCD}(K, \infty)$ condition.

The $\text{RCD}^*(K, N)$ condition is stable under the pmG convergence.

Theorem 2.10 ([26]) (**Stability of $\text{RCD}^*(K, N)$**)

Let \mathcal{X}_n be a sequence of p.m.m. spaces satisfying the $\text{RCD}^*(K, N)$ condition. If \mathcal{X}_n converges to \mathcal{X}_∞ in the pmG sense, then \mathcal{X}_∞ also satisfies the $\text{RCD}^*(K, N)$ condition.

2.5 L^2 -convergence of heat semigroups under the pmG convergence

In Gigli–Mondino–Savaré [32], they introduced L^2 -convergences on varying metric measure spaces and showed a convergence of heat semigroups in this sense under the pmG convergence of the underlying spaces with the $\text{RCD}(K, \infty)$ condition. We recall their results briefly.

Definition 2.11 (See [32, Definition 6.1]) Let $(X_n, d_n, m_n, \bar{x}_n)$ a sequence of p.m.m. spaces. Assume that $(X_n, d_n, m_n, \bar{x}_n)$ converges to $(X_\infty, d_\infty, m_\infty, \bar{x}_\infty)$ in the pmG sense. Let (X, d) be a complete separable metric space and $\iota_n : \text{supp}[m_n] \rightarrow X$ be isometries as in Definition 2.1. We identify (X_n, d_n, m_n) with $(\iota_n(X_n), d, \iota_{n\#}m_n)$ and omit ι_n .

(i) We say that $u_n \in L^2(X, m_n)$ converges weakly to $u_\infty \in L^2(X, m_\infty)$ if the following hold:

$$\sup_{n \in \mathbb{N}} \int |u_n|^2 dm_n < \infty \quad \text{and} \quad \int \phi u_n dm_n \rightarrow \int \phi u_\infty dm_\infty \quad \forall \phi \in C_{bs}(X),$$

whereby recall that $C_{bs}(X)$ denotes the set of bounded continuous functions with bounded support.

(ii) We say that $u_n \in L^2(X, m_n)$ converges strongly to $u_\infty \in L^2(X, m_\infty)$ if u_n converges weakly to u_∞ and the following holds:

$$\limsup_{n \rightarrow \infty} \int |u_n|^2 dm_n \leq \int |u_\infty|^2 dm_\infty.$$

Let $\{H_t^n\}_{t \geq 0}$ be the $L^2(X, m_n)$ -semigroup corresponding to the Cheeger energy Ch_n . Then the following theorem states that $\{H_t^n\}_{t \geq 0}$ convergence strongly in L^2 under the pmG convergence of the underlying spaces.

Theorem 2.12 (See [32, Theorem 6.11]) Let $(X_n, d_n, m_n, \bar{x}_n)$ a sequence of p.m.m. spaces satisfying the $\text{RCD}(K, \infty)$ for all $n \in \mathbb{N}$. Then, for any $u_n \in L^2(X, m_n)$ converging strongly to $u_\infty \in L^2(X, m_\infty)$, we have, for any $t > 0$

$$H_t^n u_n \text{ converges strongly to } H_t^\infty u_\infty \text{ in the sense of Definition 2.11}.$$

Note that, in [32, Theorem 6.11], the above Theorem 2.12 was stated without the condition of the infinitesimal Hilbertian.

3 Brownian Motion on RCD spaces

3.1 Brownian Motions on $\text{RCD}(K, \infty)$ Spaces

Let (X, d, m) satisfy the $\text{RCD}(K, \infty)$ condition. Let δ_x denote the unit mass at $x \in X$, and define a kernel $p(t, x, dy)$ by the action of the heat flow (see Subsection 2.4.4)

$$p(t, x, dy) := \mathcal{H}_t(\delta_x) \quad \forall t > 0, x \in X. \quad (3.1)$$

Then we have that (see [7, 4])

$$p(t, x, dy) \text{ is absolutely continuous with respect to } m \text{ for any } t > 0,$$

and we denote the density by $p(t, x, y)$. By [7, Theorem 6.1] and [4] (for the case of σ -finite reference measures), the density $p(t, x, y)$ is symmetric (i.e. $p(t, x, y) = p(t, y, x)$ for any

$x, y \in \text{supp}[m]$) and satisfies the Chapman–Kolmogorov formula. Moreover, the following action of semigroup $\{P_t\}_{t \geq 0}$

$$P_t f(x) := \int_X f(y) d\mathcal{H}_t(\delta_x)(dy) \quad (3.2)$$

is a version of the linear heat semigroup $\{H_t\}_{t \geq 0}$ defined as the gradient flow of the Cheeger energy Ch (see Subsection 2.4.5) for any $f \in L^2(X; m)$. Furthermore P_t is an extension of H_t to a continuous contraction semigroup in $L^1(X; m)$ which is *point-wise everywhere defined* on $\text{supp}[m]$ if $f \in L^\infty(X; m)$ since $P_t f$ becomes Lipschitz continuous on $\text{supp}[m]$ whenever $f \in L^\infty(X; m)$ (see [7, Theorem 6.5] and [4, Theorem 7.3]). We call $p(t, x, dy)$ and $p(t, x, y)$ the *heat kernel* and the *heat kernel density*, respectively. By the Kolmogorov extension theorem, we can construct a family of probability measures $\{\mathbb{P}^x\}_{x \in X}$ on $X^{[0, \infty)}$ and a system of Markov processes $(\{\mathbb{P}^x\}_{x \in X}, \{X_t\}_{t \geq 0})$ on X with respect to $p(t, x, dy)$.

On the other hand, we can define a Dirichlet form (i.e., a symmetric closed Markovian bilinear form) $(\mathcal{E}, \mathcal{F})$ induced by the Cheeger energy Ch as follows:

$$\mathcal{E}(u, v) = \frac{1}{4}(\text{Ch}(u + v) - \text{Ch}(u - v)) \quad u, v \in \mathcal{F} = W^{1,2}(X, d, m). \quad (3.3)$$

By [6, Lemma 6.7] (see [4, Theorem 7.2] for σ -finite reference measures), the form $(\mathcal{E}, \mathcal{F})$ becomes a quasi-regular conservative strongly-local symmetric Dirichlet form below. See [7, Proposition 4.11] for the strong locality, and the conservativeness follows from the volume growth estimate (1.2) and Sturm’s conservativeness test [65, Theorem 4].

Therefore, by [51, Theorem IV 3.5, V1.5], there exists a family of probability measures $\{\mathbb{Q}^x\}_{x \in X}$ on $C([0, \infty); X)$ and a system of conservative diffusion processes $(\{\mathbb{Q}^x\}_{x \in X}, \{Y_t\}_{t \geq 0})$ so that

$$\mathbb{E}_{\mathbb{Q}}^x(f(Y_t)) = H_t f, \quad \forall f \in L^2(X; m) \cap \mathcal{B}_b(X), \quad \forall t \geq 0, \quad \forall x \in X \setminus \mathcal{N}.$$

Here $\mathbb{E}_{\mathbb{Q}}^x$ denotes the expectation with respect to \mathbb{Q}^x and \mathcal{N} is a set of zero-capacity (so especially $m(\mathcal{N}) = 0$) with respect to the Cheeger energy $(\text{Ch}, W^{1,2}(X, d, m))$. Such systems of Markov processes are unique up to null-capacity sets.

Since $\{P_t\}_{t \geq 0}$ is a version of $\{H_t\}_{t \geq 0}$, the systems of Markov process $(\{\mathbb{P}^x\}_{x \in X}, \{X_t\}_{t \geq 0})$ and $(\{\mathbb{Q}^x\}_{x \in X}, \{Y_t\}_{t \geq 0})$ coincide except on zero-capacity sets. By considering that these two processes are equivalent, we call it a *system of Brownian motions*. To remove the ambiguity of starting points, we adopt $(\{\mathbb{P}^x\}_{x \in X}, \{X_t\}_{t \geq 0})$ for representing a system of Brownian motion (note that $(\{\mathbb{P}^x\}_{x \in X}, \{X_t\}_{t \geq 0})$ is defined at every starting point by the Kolmogorov extension theorem).

By the same argument of [10, (c) in Theorem 1.2], we have that

$$\mathbb{P}^x(C((0, \infty))) = 1 \quad \text{for every } x \in X \text{ (not only quasi-every } x \in X). \quad (3.4)$$

Note that by the conservativeness of the Dirichlet form $(\mathcal{E}, \mathcal{F})$, we know that $\mathbb{P}^x(C([0, \infty))) = 1$ for *quasi-every* $x \in X$. However, the property (3.4) is not necessarily true for general conservative quasi-regular strongly local Dirichlet form, and this property is due to the absolute continuity of the heat kernel $p(t, x, dy)$ with respect to m for any $t > 0$.

Remark 3.1 The diffusion process defined above is conventionally called *Brownian motion* ([7]), but this may indicate other diffusion processes than the standard Brownian motion in some situations. For instance, when we take $(X, d, m) = (\mathbb{R}^d, \|\cdot\|_2, \frac{1}{(2\pi)^{d/2}} \exp\{-\frac{1}{2}\|x\|_2^2\} dx)$

whereby dx denotes the Lebesgue measure, and $\|\cdot\|_2$ denotes the Euclidean distance. Then (X, d, m) satisfies $\text{RCD}(0, \infty)$ and the diffusion induced by the Cheeger energy coincides with what is known as the Ornstein-Uhlenbeck process, which is different from the standard Brownian motion on \mathbb{R}^d .

3.2 Brownian Motions on $\text{RCD}^*(K, N)$ Spaces

In this subsection, we state several properties of the Brownian motions on $\text{RCD}^*(K, N)$ spaces with bounded diameters, which do not necessarily hold only for the $\text{RCD}(K, \infty)$ condition. The main points are that the Brownian motion under Assumption 1.8 satisfies a uniform Gaussian estimate and becomes a Feller process, which implies the uniqueness of Brownian motions with respect to starting points.

Proposition 3.2 *Under Assumption 1.8, there exists a modification semigroup $\{P_t\}_{t \geq 0}$ of the heat semigroup $\{H_t\}_{t \geq 0}$ so that $\{P_t\}_{t \geq 0}$ becomes a Feller semigroup, i.e.,*

(F-1) *For any $f \in C_b(X)$, $P_t f \in C_b(X)$ for any $t > 0$.*

(F-2) *For any $f \in C_b(X)$, $\|P_t f - f\|_\infty \rightarrow 0$ as $t \downarrow 0$.*

Remark 3.3 Although the proof might be already known in some literature, we could not find good references and we give the proof for the sake of reader's convenience.

Proof. The condition (F-1) follows directly from [7, (iii) in Theorem 6.1]. We show the condition (F-2). By the generalized Bishop–Gromov inequality [26, Proposition 3.9], we have the following volume growth estimate: there exist positive constants $\nu = \nu(N, K, D) > 0$ and $c = c(N, K, D) > 0$ such that, for all $n \in \mathbb{N}$

$$m_n(B_r(x)) \geq cr^{2\nu} \quad (0 \leq r \leq 1 \wedge D). \quad (3.5)$$

In fact, taking $B_R(x_0) = B_D(x_0)$ for some $x_0 \in X$, we have

$$m_n(B_r(x)) \geq \frac{\int_0^r \Theta_{K/N}(t)^N dt}{\int_0^D \Theta_{K/N}(t)^N dt} m_n(B_D(x)) = c(N, K, D) \int_0^r \Theta_{K/N}(t)^N dt.$$

Here we used $m_n(X_n) = 1$ and $c(N, K, D) = \frac{1}{\int_0^D \Theta_{K/N}(t)^N dt}$. Thus we have (3.5).

On the other hand, under Assumption 1.8, the volume doubling property ([68]) and the Poincaré inequality ([35, 59, 60]) hold. According to [66, 67] (note that the MCP condition is satisfied under Assumption 1.8), we have the following Gaussian heat kernel estimate: There exist positive constants $C_1 = C_1(N, K, D)$, $C'_1 = C'_1(N, K, D)$, $C_2 = C_2(N, K, D)$ and $C'_2 = C'_2(N, K, D)$ depending only on N, K, D so that

$$\frac{C'_1}{m(B_{\sqrt{t}}(x))} \exp\left\{-C'_2 \frac{d(x, y)^2}{t}\right\} \leq p(t, x, y) \leq \frac{C_1}{m(B_{\sqrt{t}}(x))} \exp\left\{-C_2 \frac{d(x, y)^2}{t}\right\}, \quad (3.6)$$

for all $x, y \in X$ and $0 < t \leq D^2$. Here the heat kernel $p(t, x, y)$ means the integral kernel of the heat semigroup $P_t f(x) = \int_X f p(t, x, y) m(dy)$ for $t > 0$. Combining with (3.5), we have the following uniform upper heat kernel estimate: (**uniform Gaussian estimate**)

$$p(t, x, y) \leq \frac{C_1}{ct^\nu} \exp\left\{-C_2 \frac{d(x, y)^2}{t}\right\}, \quad (3.7)$$

for all $x, y \in X$ and $0 < t \leq D^2$. Here constants C_1, C_2, c, ν only depends on the given constants N, K, D .

For given $\varepsilon > 0$, take $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $d(x, y) < \delta$. By the Gaussian estimate (3.7), we can choose a positive number T such that $p(t, x, y) < \varepsilon$ for any $0 < t < T$ and any $x, y \in X$ satisfying $d(x, y) \geq \delta$. Then we have that, for any $x \in X$

$$\begin{aligned} |Pf(x) - f(x)| &= \left| \int_X p(t, x, y) f(y) m(dy) - f(x) \right| \\ &\leq \int_X p(t, x, y) |f(y) - f(x)| m(dy) \\ &= \int_{B_\delta(x)} p(t, x, y) |f(y) - f(x)| m(dy) + \int_{X \setminus B_\delta(x)} p(t, x, y) |f(y) - f(x)| m(dy) \\ &\leq \varepsilon + 2\varepsilon \|f\|_\infty. \end{aligned}$$

Thus we have shown that (F-2) holds. \square

4 Examples

In this section, several specific examples satisfying Assumption 1.1 or Assumption 1.8 are given. In the first subsection, we explain weighted Riemannian manifolds whose weighted Ricci curvature is bounded below, and its pmG limit spaces. In the second subsection, we explain Alexandrov spaces, which is roughly speaking, the sectional curvature is bounded flow below. In the third subsection, we give Hilbert spaces with log-concave probability measures.

4.1 Weighted Riemannian Manifolds and pmG Limit Spaces

Let $(M_n, g_n, w_n, \bar{x}_n)$ be a sequence of pointed complete and connected weighted N -dimensional Riemannian manifolds whose weight satisfies $w_n = e^{-V_n}$ for a twice continuously differentiable function $V_n \in C^2(M_n)$. We write the corresponding pointed metric measure space $\mathcal{M}_n = (M_n, d_{g_n}, w_n \text{Vol}_n, \bar{x}_n)$ whereby d_{g_n} denotes the distance function associated with the Riemannian metric g_n ; Vol_n denotes the Riemannian volume measure; and $\bar{x}_n \in M_n$ is a fixed point. Let the weighted Ricci curvature $\text{Ric}_{\mathcal{M}_n}$ of \mathcal{M}_n be bounded from below by K : there exists $K \in \mathbb{R}$ so that

$$\text{Ric}_{\mathcal{M}_n} = \text{Ric}_{g_n} + \nabla^2 V_n \geq K g_n,$$

whereby Ric_{g_n} means the Ricci curvature of (M_n, g_n) and ∇^2 means the Hessian. Then \mathcal{M}_n satisfies $\text{RCD}(K, \infty)$ spaces ([62, 68]). Even when $V_n : M_n \rightarrow \mathbb{R}$ is not in $C^2(M_n)$, if $\text{Ric}_{g_n} \geq K$ and

$$V_n : M_n \rightarrow \mathbb{R} \text{ is } K'\text{-convex (see [68])},$$

then \mathcal{M}_n satisfies $\text{RCD}(K + K', \infty)$. If, moreover,

$$V_n : M_n \rightarrow \mathbb{R} \text{ is } (K', N')\text{-convex (see [26])},$$

then \mathcal{M}_n satisfies $\text{RCD}^*(K + K', N + N')$. The Brownian motion on M_n is a Markov process whose infinitesimal generator A_n is

$$A_n = \frac{1}{2} \Delta_{M_n} - \langle \nabla V_n, \nabla \rangle,$$

whereby Δ_{M_n} is the Laplace-Beltrami operator on M_n .

If \mathcal{M}_n satisfying $\text{RCD}(K, \infty)$ (or, $\text{RCD}^*(K, N)$) converges to a metric measure space \mathcal{M}_∞ in pmG sense, then the limit space \mathcal{M}_∞ satisfies $\text{RCD}(K, \infty)$ (or, $\text{RCD}^*(K, N)$), respectively (see [32, 26]). Thus we can apply our main results (Theorem 1.2, or Theorem 1.7 for the case of $\text{RCD}^*(K, N)$) to Brownian motions on these spaces and we can obtain the weak convergence of the Brownian motions.

We have various singular examples appearing as the limit space. See e.g., [22, Example 8]. We give one of the simplest examples included in this framework.

Example 4.1 (Collapsing: Torus \rightarrow Circle)

Let $\mathbb{S}^1 \subset \mathbb{R}^2$ be the unit circle. Let $d_{\mathbb{S}^1}$ be the intrinsic distance on \mathbb{S}^1 , that is, the distance between x and y is defined by the infimum over lengths of geodesics on \mathbb{S}^1 connecting x and y . Let

$$\overline{H}_{\mathbb{S}^1} := \frac{1}{H_{\mathbb{S}^1}(\mathbb{S}^1)} H_{\mathbb{S}^1}$$

be the normalized Hausdorff measure on $(\mathbb{S}^1, d_{\mathbb{S}^1})$. Let $\mathbb{T}_n = \mathbb{S}^1 \times \mathbb{S}^1$ be a two-dimensional flat torus with a metric $d_n = d_{\mathbb{S}^1} \otimes \frac{1}{n} d_{\mathbb{S}^1}$ and the normalized Hausdorff measure \overline{H}_n on (\mathbb{T}_n, d_n) , whereby

$$d_{\mathbb{S}^1} \otimes \frac{1}{n} d_{\mathbb{S}^1}((x_1, y_1), (x_2, y_2)) := \sqrt{d_{\mathbb{S}^1}^2(x_1, x_2) + \frac{1}{n^2} d_{\mathbb{S}^1}^2(y_1, y_2)}.$$

Then $(\mathbb{T}_n, d_n, \overline{H}_n)$ satisfies the $\text{RCD}^*(0, 2)$ for any $n \in \mathbb{N}$ and converges to $(\mathbb{S}^1, d_{\mathbb{S}^1}, \overline{H}_{\mathbb{S}^1})$ in the measured Gromov sense. Thus we can apply our result (Theorem 1.9) and the weak convergence of the Brownian motions on M_n is equivalent to the pmG convergence of M_n .

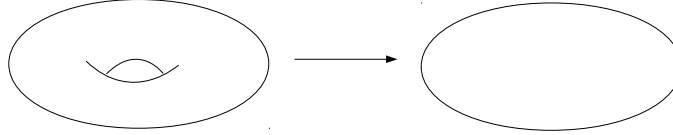


Figure 1: Tori Converge to a Circle.

4.2 Alexandrov Spaces

We explain Alexandrov spaces, which are generalization of lower bounds of sectional curvatures for metric spaces. We refer a reader to [20] for basic theory of Alexandrov spaces. Let (X, d) be a locally compact length space. For a triple of points $p, q, r \in X$, a geodesic triangle $\triangle pqr$ is a triplet of geodesics joining each two points. Let $M^N(K)$ be the N -dimensional complete simply connected space of constant sectional curvature K . For a geodesic triangle $\triangle pqr$, we denote by $\triangle \widetilde{pqr}$ a geodesic triangle in $M^2(K)$ whose corresponding edges have the same lengths as $\triangle pqr$.

A locally compact length space (X, d) is said to be an *Alexandrov space with $\text{Curv} \geq K$* if for every point $x \in X$, there exists an open set U_x including x so that for every geodesic triangle $\triangle pqr$ whose edges are totally included in U_x , the corresponding geodesic triangle

$\triangle \widetilde{p}\widetilde{q}\widetilde{r}$ satisfies the following condition: for every point $z \in qr$ and $\widetilde{z} \in \widetilde{q}\widetilde{r}$ with $d(q, z) = d(\widetilde{q}, \widetilde{z})$, we have

$$d(p, z) \geq d(\widetilde{p}, \widetilde{z}).$$

If we consider a complete N -dimensional Riemannian manifold (M, g) , then (M, g) is an Alexandrov space with $\text{Curv} \geq K$ if and only if $\sec(M) \geq K$, whereby $\sec(M)$ means the sectional curvature of M .

Let $\mathcal{X} = (X, d, H)$ be an N -dimensional Alexandrov space with $\text{Curv} \geq K$ and H be the Hausdorff measure (see e.g., [20] for details). According to [57, 74], \mathcal{X} satisfies $\text{CD}^*((N-1)K, N)$. Moreover, as was shown in [46], \mathcal{X} satisfies the infinitesimal Hilbertian condition, and as a result, \mathcal{X} satisfies $\text{RCD}^*((N-1)K, N)$. Thus we can apply our results (Theorem 1.2, 1.4) and if a sequence of pointed Alexandrov spaces X_n with $\text{Curv} \geq K$ converges to the limit space X_∞ in the pmG sense, then the Brownian motions on X_n converge weakly to the limit Brownian motion on X_∞ . We give several examples.

Example 4.2 (Cone \rightarrow Interval)

Let $X \subset \mathbb{R}^3$ be a cone defined by $X_n = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = \frac{1}{n}x, 0 \leq x < 1\} \cup \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = \frac{1}{n}, x = 1\}$. Let d_n be the restriction of the Euclidean distance on X_n and \overline{H}_n be the normalized Hausdorff measure on X_n . Then $(X_n, d_n, \overline{H}_n)$ satisfies $\text{RCD}^*(0, 2)$ for large $n \in \mathbb{N}$ and converges to $([0, 1], |\cdot|, m)$ in the measured Gromov sense, whereby m is some measure on $[0, 1]$. Thus we can apply our result (Theorem 1.9) and the weak convergence of the Brownian motions on M_n is equivalent to the pmG convergence of M_n .

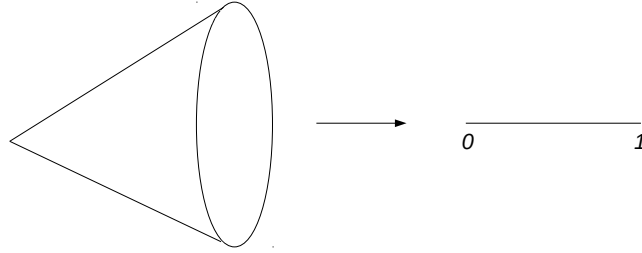


Figure 2: Cones Converge to an Interval.

As a second example, we give a sequence of polygons made by the barycentric subdivision. The limit space has dense singularities.

Example 4.3 (Dense Singularities) [[55, Examples. (2)]]

Let $M_1 = (M_1, d_1, H_1)$ be a polyhedron in \mathbb{R}^3 with the geodesic metric d_1 and the Hausdorff measure. Then we can check that M_1 is an Alexandrov space with $\text{Curv} \geq 0$, which is also an $\text{RCD}^*(0, 2)$ space. For any vertices $p \in M_1$, let $\angle(M_1, p)$ denote the sum of all inner angles of at p of faces T 's such that p is a vertex of T . Now we construct a sequence of polyhedra $\{M_n\}_{n \in \mathbb{N}}$ inductively. Let M_1 be a tetrahedron in \mathbb{R}^3 with the barycenter o . Let M_n be defined. Then we define M_{n+1} as follows: Take a monotone decreasing sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ such that $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ with $0 < \varepsilon_n < 1$ and $\varepsilon := \prod_{n=1}^{\infty} (1 - \varepsilon_n) > 0$. We take the barycentric subdivision of M_n . Keep the original vertices in M_n and move the new vertices generated by the barycentric subdivision outward along rays emanating from o so small that, for the new polyhedra M_{n+1} generated by the new and original vertices, we have

$$2\pi - \angle(M_{n+1}, p) \geq (1 - \varepsilon_n)(2\pi - \angle(M_n, p)),$$

for any vertex $p \in M_n$. Then there exists the Hausdorff-limit of $M_n = (M_n, d_n, H_n)$, denoted by M_∞ . The limit space M_∞ is a two-dimensional Alexandrov space with nonnegative curvature. In particular, $(M_n, d_n, \overline{H}_n)$ satisfies the $\text{RCD}^*(0, 2)$ and converges to $(M_\infty, d_\infty, \overline{H}_\infty)$ in the measured Gromov sense. The limit space M_∞ also satisfies the $\text{RCD}^*(0, 2)$ by the stability of $\text{RCD}^*(K, N)$ spaces under the measured Gromov convergence (see [26]). The set of singular points is dense in M_∞ (see [55]). Thus we can apply our result (Theorem 1.9) and the weak convergence of the Brownian motions on M_n is equivalent to the pmG convergence of M_n .

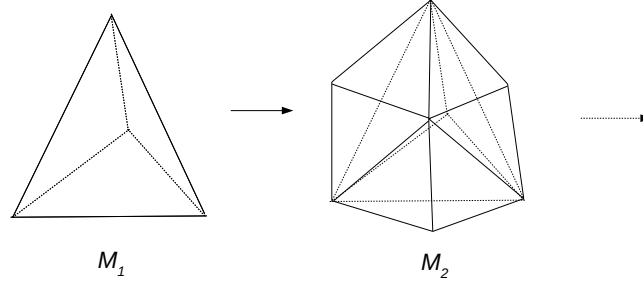


Figure 3: Polyhedra Generated by Barycentric Subdivision.

4.3 Hilbert Space with Log-concave Measures

In this subsection, we give a specific class of $\text{RCD}(0, \infty)$ spaces, which is a Hilbert space with log-concave measures. This case is due to [10].

Let H be a separable Hilbert space, which would be a finite- or infinite-dimensional space, with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \|$. A Borel probability measure γ on H satisfies *log-concave condition* if, for all pairs of open subsets $A, B \subset H$, it holds that

$$\log((1-t)A + tB) \geq (1-t) \log \gamma(A) + t \log \gamma(B), \quad \forall t \in [0, 1].$$

Let $K = \text{supp}[\gamma]$ and $A = A(\gamma)$ be the smallest closed linear subspace containing K . We write canonically

$$A = H_0 + h_0, \quad h_0 \in K, \quad \|h_0\| \leq \|k\|, \quad \forall k \in K,$$

so that h_0 is the element of the minimal norm in K and H_0 is a closed linear subspace in H .

Let $C_b^1(A)$ be the set of all $\Phi : A \rightarrow \mathbb{R}$ which are bounded, continuous and Fréchet differentiable with bounded continuous gradient $\nabla \Phi : A \rightarrow H_0$. Then, according to [10, Theorem 1.2], the following bilinear form becomes closable and the closed form becomes a symmetric quasi-regular Dirichlet form $\mathcal{E} = \mathcal{E}_{\|\cdot\|, \gamma}$:

$$\mathcal{E}(u, v) = \int_K \langle \nabla u, \nabla v \rangle_{H_0} d\gamma, \quad u, v \in \mathcal{F} := \overline{C_b^1(A)}^{\sqrt{\mathcal{E} + \|\cdot\|_2^2}} \quad (4.1)$$

In [10], the corresponding semigroup $\{P_t\}_{t \geq 0}$ associated with $(\mathcal{E}, \mathcal{F})$ satisfies EVI_0 property, which is equivalent to the $\text{RCD}(0, \infty)$ condition of $(H, \|\cdot\|, \gamma)$ according to [7]. Let $H_n = (H, \|\cdot\|_n, \gamma_n, \bar{x}_n)$ be a sequence of pointed Hilbert spaces with log-concave probability measures satisfying the above conditions. Then we can apply our results (Theorem 1.2, 1.4) and the weak convergence of the Brownian motions on H_n to that on H_∞ follows from the pmG convergence of the underlying spaces H_n to H_∞ .

Various infinite dimensional examples are included in the framework of Hilbert spaces with log-concave probability measures. For instance, all measures γ of the following form satisfies the log-concave condition: let dx be the Lebsgue measure on \mathbb{R}^N and

$$\gamma = \frac{1}{Z} e^{-V} dx, \quad \text{whereby } V : H = \mathbb{R}^N \rightarrow \mathbb{R} \text{ convex and } Z = \int_{\mathbb{R}^N} e^{-V} dx < +\infty, \quad (4.2)$$

such as all Gaussian measures and all Gibbs measures on on a finite lattice with a convex Hamiltonian. See [10, Section 1.2] for various infinite-dimensional literatures related to stochastic partial differential equations. We give several finite-dimensional examples.

Example 4.4 [[10]] We give several examples associated with stochastic differential equations (SDE). We first give an SDE on the Euclidean space \mathbb{R}^N with variable potentials. We secondly give an SDE on a variable convex domain in \mathbb{R}^N with variable potentials.

- (a) **(SDE with variable convex potentials)** Let $H = \mathbb{R}^N$ with $1 < N < \infty$. Let $V_n : \mathbb{R}^N \rightarrow \mathbb{R}$ be a sequence of convex functionals with a Lipschitz continuous gradients $\nabla V_n : \mathbb{R}^N \rightarrow \mathbb{R}^N$ and $\int_{\mathbb{R}^N} e^{-V_n} dx < \infty$. Take

$$\gamma_n = \frac{1}{Z_n} e^{-V_n} dx, \quad \text{whereby } Z_n = \int_{\mathbb{R}^N} e^{-V_n} dx.$$

Then γ_n becomes a log-concave probability measure. Then the diffusion process associated with the Dirichlet form \mathcal{E}_n in (4.1) is a solution of the following SDE:

$$dX_t^n = -\nabla V_n(X_t^n) dt + \sqrt{2} dW_t, \quad X_0 = \bar{x}_n.$$

If, V_n converges to V_∞ uniformly and $\bar{x}_n \rightarrow \bar{x}_\infty$, then it is easy to check that γ_n converges to γ_∞ weakly and $(\mathbb{R}^N, \|\cdot\|_2, \gamma_n, \bar{x}_n)$ converges to $(\mathbb{R}^N, \|\cdot\|_2, \gamma_\infty, \bar{x}_\infty)$ in the pmG sense. Thus we can apply our results (Theorem 1.2, 1.4) and the Brownian motions on $H_n = (H, \|\cdot\|_n, \gamma_n, \bar{x}_n)$ converges weakly to the limit Brownian motion on H_∞ .

- (b) **(SDE on variable convex subsets with variable convex potentials)** Let $H = \mathbb{R}^N$ with $1 < N < \infty$ and $U_n \subset \mathbb{R}^N$ be a convex open set. We consider a convex functional $V_n \in C^{1,1}(U_n)$ and $V_n \equiv +\infty$ on $\mathbb{R}^N \setminus U_n$ with $\int_{U_n} e^{-V_n} dx < \infty$. Take

$$\gamma_n = \frac{1}{Z_n} e^{-V_n} dx|_{U_n}, \quad \text{whereby } Z_n = \int_{U_n} e^{-V_n} dx.$$

Then γ_n becomes a log-concave probability measure. Then the diffusion process associated with the Dirichlet form \mathcal{E}_n in (4.1) is a solution of the following SDE:

$$dX_t^n = -\nabla V_n(X_t^n) dt + \sqrt{2} dW_t + \mathbf{n}(X_t) dL_t^n, \quad X_0 = \bar{x}_n \in U. \quad (4.3)$$

Here \mathbf{n} is an inner normal vector to ∂U_n and L^n is a continuous monotone non-decreasing process which increases only when $X_t \in \partial U_n$.

If the closure \bar{U}_n converges to a closed convex subset $\bar{U}_\infty \subset \mathbb{R}^N$ in the Hausdorff sense (see e.g., [34]), $\bar{x}_n \rightarrow \bar{x}_\infty$ and V_n converges to V_∞ uniformly, then it is easy to check that γ_n converges to γ_∞ weakly and $(\mathbb{R}^N, \|\cdot\|_2, \gamma_n, \bar{x}_n)$ converges to $(\mathbb{R}^N, \|\cdot\|_2, \gamma_\infty, \bar{x}_\infty)$ in the pmG sense. Thus we can apply our results (Theorem 1.2, 1.4) and the diffusion processes associated with (4.3) on $H_n = (H, \|\cdot\|_n, \gamma_n, \bar{x}_n)$ converges weakly to the solution of the limit SDE on H_∞ .

5 Proof of Theorem 1.2

We first show the implication of (ii) \implies (i) in Theorem 1.2.

Proof of (ii) \implies (i) in Theorem 1.2. If we assume (ii), then it is obvious that the initial distributions \tilde{m}_n converge weakly to \tilde{m}_∞ . Since the weak convergence of \tilde{m}_n to \tilde{m}_∞ is equivalent to the convergence of m_n to m_∞ in the sense of (2.2) (easy to check), we finish the proof of the implication (ii) \implies (i) in Theorem 1.2. \square

We now show the implication (i) \implies (ii).

Proof of (i) \implies (ii) in Theorem 1.2. By Definition 2.1, there exist a complete separable metric space (X, d) and a family of isometric embeddings $\iota_n : X_n \rightarrow X$ such that, for any bounded continuous function $f : X \rightarrow \mathbb{R}$ with bounded support, we have

$$\int_X f d(\iota_n \# m_n) \rightarrow \int_X f d(\iota_\infty \# m_\infty). \quad (5.1)$$

Set the notation for the laws of Brownian motions as follows:

$$\mathbb{B}_n^{\tilde{m}_n} := (\iota_n(B^n), \mathbb{P}_n^{\tilde{m}_n}), \quad \mathbb{B}_n^{\bar{x}_n} := (\iota_n(B^n), \mathbb{P}_n^{\bar{x}_n}).$$

Hereafter we identify $\iota_n(X_n)$ with X_n and we omit ι_n for simplifying the notation.

To show the weak convergence of the Brownian motions, we have two steps. The first is to show the weak convergence of finite-dimensional distributions, and the second is to show tightness. We first show the weak convergence of finite-dimensional distributions in the case that the initial distribution is the Dirac measure $\delta_{\bar{x}_n}$.

Lemma 5.1 (Convergence of Finite-Dimensional Distributions) *For any $k \in \mathbb{N}$, $0 = t_0 < t_1 < t_2 < \dots < t_k < \infty$ and $f_1, f_2, \dots, f_k \in C_b(X)$, the following holds:*

$$\mathbb{E}^{\bar{x}_n}[f_1(B_{t_1}^n) \dots f_k(B_{t_k}^n)] \xrightarrow{n \rightarrow \infty} \mathbb{E}^{\bar{x}_\infty}[f_1(B_{t_1}^\infty) \dots f_k(B_{t_k}^\infty)]. \quad (5.2)$$

Proof. Since the limit Brownian motion $\mathbb{B}_\infty^{\bar{x}_\infty}$ is conservative, it suffices to show the statement only for $f_1, f_2, \dots, f_k \in C_b(X) \cap L^2(X; m_\infty)$. In fact, for any $\varepsilon > 0$ and $T > 0$, there exists $R = R(\varepsilon, T)$ so that the open ball $B_R(\bar{x}_\infty)$ satisfies

$$\mathbb{E}^{\bar{x}_\infty} \mathbf{1}_{B_R(\bar{x}_\infty)}(B_t^\infty) = \mathbb{P}^{\bar{x}_\infty}(B_t^\infty \in B_R(\bar{x}_\infty)) \geq 1 - \varepsilon \quad \forall t \in [0, T].$$

If we know that $\mathbb{E}^{\bar{x}_n}(f(B_t^n))$ converges to $\mathbb{E}^{\bar{x}_\infty}(f(B_t^\infty))$ for any $f \in C_b(X) \cap L^2(X; m_\infty)$, then we know that

$$\lim_{n \rightarrow \infty} \mathbb{P}^{\bar{x}_n}(B_t^n \in B_R(\bar{x}_\infty)) = \lim_{n \rightarrow \infty} \mathbb{E}^{\bar{x}_n}(\mathbf{1}_{B_R(\bar{x}_\infty)}(B_t^n)) = \mathbb{E}^{\bar{x}_\infty}(\mathbf{1}_{B_R(\bar{x}_\infty)}(B_t^\infty)) \geq 1 - \varepsilon \quad \forall t \in [0, T].$$

Therefore, for any $f_1, \dots, f_k \in C_b(X)$, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{E}^{\bar{x}_n}(f_1(B_{t_1}^n) \dots f_k(B_{t_k}^n)) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}^{\bar{x}_n}\left(f_1(B_{t_1}^n) \dots f_k(B_{t_k}^n) : \bigcap_{j=1}^k \{B_{t_j}^n \in B_R(\bar{x}_\infty)\}\right) \\ & \quad + \lim_{n \rightarrow \infty} \mathbb{E}^{\bar{x}_n}\left(f_1(B_{t_1}^n) \dots f_k(B_{t_k}^n) : \left(\bigcap_{j=1}^k \{B_{t_j}^n \in B_R(\bar{x}_\infty)\}\right)^c\right) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}^{\bar{x}_n}\left(f_1 \mathbf{1}_{B_R}(\bar{x}_\infty)(B_{t_1}^n) \dots f_k \mathbf{1}_{B_R}(\bar{x}_\infty)(B_{t_k}^n)\right). \end{aligned}$$

Thus we may show the proof only for $f_1, f_2, \dots, f_k \in C_b(X) \cap L^2(X; m_\infty)$.

Recall that we have the following equality (see Subsection 3.1): for every $f \in C_b(X) \cap L^2(X; m_\infty)$,

$$\mathbb{E}_n^x(f(B_t^n)) = P_t^n f(x), \quad (5.3)$$

for every $x \in X_n$. Here recall that $\{P_t^n\}_{t \geq 0}$ is the semigroup defined in (3.2) by the action of the heat flow whereby P_t is a modification of the heat semigroup H_t and $P_t^n f(x)$ can be defined for every point $x \in X^n$ if $f \in C_b(X) \cap L^2(X; m_\infty)$. Since the Brownian motion $(\{\mathbb{P}_n^x\}_{x \in X_n}, \{B_t^n\}_{t \geq 0})$ is constructed by the Kolmogorov extension theorem with the integral kernel $p_n(t, x, dy)$ of $\{P_t^n\}_{t \geq 0}$, the equality (5.3) holds for every point $x \in X_n$.

By using the Markov property, for all $n \in \bar{\mathbb{N}}$, we have

$$\begin{aligned} & \mathbb{E}_n^{\bar{x}_n}[f_1(B_{t_1}^n) \cdots f_k(B_{t_k}^n)] \\ &= P_{t_1-t_0}^n \left(f_1 P_{t_2-t_1}^n \left(f_2 \cdots P_{t_k-t_{k-1}}^n f_k \right) \right) (\bar{x}_n) \\ &=: \mathcal{P}_k^n(\bar{x}_n). \end{aligned}$$

By [4, Theorem 7.3], \mathcal{P}_k^n is bounded Lipschitz on X_n whose Lipschitz constant depends only on the curvature lower-bound K .

For later arguments, we extend \mathcal{P}_k^n to the whole space X (note that \mathcal{P}_k^n is defined only on each X_n). The key point is to extend \mathcal{P}_k^n to the whole space X preserving its Lipschitz regularity and bounds.

Proposition 5.2 (McShane Extension [52, Corollary 1,2]) *Let $\widetilde{\mathcal{P}}_k^n$ be the following function on the whole space X (which is called McShane extension [52]):*

$$\widetilde{\mathcal{P}}_k^n(x) := \left(\sup_{a \in X_n} \{ \mathcal{P}_k^n(a) - Hd(a, x) \} \wedge \sup_{a \in X_n} \mathcal{P}_k^n(a) \right) \vee \inf_{a \in X_n} \mathcal{P}_k^n(a) \quad x \in X, \quad (5.4)$$

whereby H is the same Lipschitz constant of the original function \mathcal{P}_k^n . Then we have that $\widetilde{\mathcal{P}}_k^n$ is a bounded Lipschitz continuous function on the whole space X with the same Lipschitz constant H and the same bound, and satisfies $\widetilde{\mathcal{P}}_k^n = \mathcal{P}_k^n$ on X_n .

Coming back to the proof of Lemma 5.1, we have that

$$\begin{aligned} & \left| \mathbb{E}_n^{\bar{x}_n}[f_1(B_{t_1}^n) \cdots f_k(B_{t_k}^n)] - \mathbb{E}_n^{\bar{x}_\infty}[f_1(B_{t_1}^\infty) \cdots f_k(B_{t_k}^\infty)] \right| \\ &= |\mathcal{P}_k^n(\bar{x}_n) - \mathcal{P}_k^\infty(\bar{x}_\infty)| \\ &\leq |\mathcal{P}_k^n(\bar{x}_n) - \widetilde{\mathcal{P}}_k^n(\bar{x}_\infty)| + |\widetilde{\mathcal{P}}_k^n(\bar{x}_\infty) - \mathcal{P}_k^\infty(\bar{x}_\infty)| \\ &=: (\text{I})_n + (\text{II})_n. \end{aligned} \quad (5.5)$$

Thus it suffices to show that $(\text{I})_n$ and $(\text{II})_n$ converge to zero as n goes to infinity.

We first show that $(\text{I})_n$ converges to zero as n goes to infinity. Since

$$\|P_t^n f\|_\infty = \|f\|_\infty \left\| \int_{X_n} p_n(t, x, y) m_n(dy) \right\|_\infty \leq \|f\|_\infty,$$

for any $f \in C_b(X_n) \cap L^2(X; m_\infty)$, we have

$$\sup_{n \in \bar{\mathbb{N}}} \|\mathcal{P}_k^n\|_\infty \leq \prod_{i=1}^k \|f_i\|_\infty < \infty. \quad (5.6)$$

Therefore, by the property of the McShane extension in Proposition 5.2, we also have that

$$\sup_{n \in \mathbb{N}} \|\widetilde{\mathcal{P}}_k^n\|_\infty < \infty. \quad (5.7)$$

By [4, Theorem 7.3], we have that $\text{Lip}_X(P_t^n f) \leq C(t, K)\|f\|_\infty$ for any $f \in L^\infty(X_n; m_n) \cap L^2(X; m_\infty)$ for some positive $C(t, K)$ depending only on t, K . Thus by considering (5.6), there exists a constant L depending only on t_k, K and $\|f_1\|_\infty, \dots, \|f_k\|_\infty$ (but independent of n) so that

$$\sup_{n \in \mathbb{N}} \text{Lip}_X(\mathcal{P}_k^n) \leq \sup_{n \in \mathbb{N}} C(t_k, K)\|f_k \mathcal{P}_{k-1}^n\|_\infty < L < \infty.$$

By the property of the McShane extension in Proposition 5.2, we have that

$$\sup_{n \in \mathbb{N}} \text{Lip}_X(\widetilde{\mathcal{P}}_k^n) \leq \sup_{n \in \mathbb{N}} C(t_k, K)\|f_k \mathcal{P}_{k-1}^n\|_\infty < L < \infty. \quad (5.8)$$

Thus we have

$$\begin{aligned} (\text{I})_n &= |\mathcal{P}_k^n(\bar{x}_n) - \widetilde{\mathcal{P}}_k^n(\bar{x}_\infty)| = |\widetilde{\mathcal{P}}_k^n(\bar{x}_n) - \widetilde{\mathcal{P}}_k^n(\bar{x}_\infty)| \\ &\leq \text{Lip}(\widetilde{\mathcal{P}}_k^n) d(\bar{x}_n, \bar{x}_\infty) \\ &\leq L d(\bar{x}_n, \bar{x}_\infty) \\ &\rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Now we show that $(\text{II})_n$ goes to zero as n tends to infinity. By the uniform boundedness (5.7) and the uniform Lipschitz continuity (5.8), we can apply the Ascoli–Arzelá theorem to $\{\widetilde{\mathcal{P}}_k^n\}_{n \in \mathbb{N}}$ so that $\{\widetilde{\mathcal{P}}_k^n\}_{n \in \mathbb{N}}$ is relatively compact, which means, for any subsequence $\{\widetilde{\mathcal{P}}_k^{n'}\}_{\{n'\} \subset \{n\}}$, there exists a further subsequence $\{\widetilde{\mathcal{P}}_k^{n''}\}_{\{n''\} \subset \{n'\}}$ satisfying

$$\widetilde{\mathcal{P}}_k^{n''} \rightarrow F'' \quad \text{uniformly in } X. \quad (5.9)$$

On the other hand, we have that \mathcal{P}_k^n converges to \mathcal{P}_k^∞ L^2 -strongly in the sense of Definition 2.11. We give a proof below.

Lemma 5.3 \mathcal{P}_k^n converges to \mathcal{P}_k^∞ in the L^2 -strong sense in Definition 2.11.

Proof. By Theorem 2.12, the statement is true for $k = 1$. Assume that the statement is true when $k = l$. By noting,

$$\mathcal{P}_{l+1}^n = P_{t_{l+1}-t_l}^n(f_{l+1}^{(n)} \mathcal{P}_l^n),$$

by Theorem 2.12, it suffices to show $f_{l+1} \mathcal{P}_l^n \rightarrow f_{l+1} \mathcal{P}_l^\infty$ strongly in L^2 . This is easy to show because $\mathcal{P}_l^n \rightarrow \mathcal{P}_l^\infty$ strongly (the assumption of the induction), $f_{l+1} \in C_b(X)$ and \mathcal{P}_l^n is bounded uniformly in n because of (5.7). Thus the statement is true for any $k \in \mathbb{N}$. \square

Coming back to the proof of Lemma 5.1.

Proof of Lemma 5.1. By using Lemma 5.3 and the uniform convergence (5.9), it is easy to see that

$$F''|_{X_\infty} = \mathcal{P}_k^\infty,$$

whereby $F''|_{X_\infty}$ means the restriction of F'' into X_∞ . The R.H.S. \mathcal{P}_k^∞ of the above equality is clearly independent of choices of subsequences and thus the limit $F''|_{X_\infty}$ is independent of choices of subsequences. Therefore we conclude that

$$\widetilde{\mathcal{P}}_k^n \rightarrow \mathcal{P}_k^\infty \quad \text{uniformly in } X_\infty. \quad (5.10)$$

Going back to showing that $(\text{II})_n$ goes to zero, we have that

$$\begin{aligned} (\text{II})_n &= |\widetilde{\mathcal{P}}_k^n(\bar{x}_\infty) - \mathcal{P}_k^\infty(\bar{x}_\infty)| \leq \|\widetilde{\mathcal{P}}_k^n - \mathcal{P}_k^\infty\|_{\infty, X_\infty} \\ &\rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Here $\|\cdot\|_{\infty, X_\infty}$ means the uniform norm on X_∞ . Thus we finish the proof of Lemma 5.1. \square

We next show the weak convergence of finite-dimensional distributions for the case that initial distributions are W_1 -convergent, which includes \tilde{m}_n for the case of $m_n(X_n) = \infty$.

Lemma 5.4 *Let $\nu_n \in \mathcal{P}(X_n)$ be a sequence of probability measures on $X_n \subset X$ converging to $\nu_\infty \in \mathcal{P}(X_\infty)$ in W_1 -distance. Then, for any $k \in \mathbb{N}$, $0 = t_0 < t_1 < t_2 < \dots < t_k < \infty$ and $f_1, f_2, \dots, f_k \in C_b(X) \cap L^2(X; m_\infty)$, the following holds:*

$$\mathbb{E}^{\nu_n}[f_1(B_{t_1}^n) \cdots f_k(B_{t_k}^n)] \xrightarrow{n \rightarrow \infty} \mathbb{E}^{\nu_\infty}[f_1(B_{t_1}^\infty) \cdots f_k(B_{t_k}^\infty)].$$

Proof. By the same argument at the beginning of Lemma 5.1, it suffices to show the statement for any $f_1, f_2, \dots, f_k \in C_b(X) \cap L^2(X; m_\infty)$. Recall that we set in Lemma 5.1 as follows:

$$\begin{aligned} &\mathbb{E}_n^x[f_1(B_{t_1}^n) \cdots f_k(B_{t_k}^n)] \\ &= P_{t_1-t_0}^n \left(f_1^{(n)} P_{t_2-t_1}^n \left(f_2^{(n)} \cdots P_{t_k-t_{k-1}}^n f_k^{(n)} \right) \right) (x) \\ &=: \mathcal{P}_k^n(x). \end{aligned}$$

By the Kantorovich–Rubinstein duality (see e.g., [73, Theorem 5.10]), we have

$$W_1(\nu_n, \nu_\infty) = \frac{1}{L} \sup \left\{ \int_X f d\nu_n - \int_X f d\nu_\infty : f \in \text{Lip}_b(X), \text{Lip}(f) \leq L \right\}. \quad (5.11)$$

According to (5.7) and (5.8), we have that $\widetilde{\mathcal{P}}_k^n$ is bounded and $\sup_{n \in \mathbb{N}} \text{Lip}(\widetilde{\mathcal{P}}_k^n) < L < \infty$ for some constant L . Thus we have that

$$\left| \int_X \widetilde{\mathcal{P}}_k^n d\nu_n - \int_X \widetilde{\mathcal{P}}_k^n d\nu_\infty \right| \leq L W_1(\nu_n, \nu_\infty). \quad (5.12)$$

Since $\widetilde{\mathcal{P}}_k^n$ converges to \mathcal{P}_k^∞ uniformly in $C_b(X_\infty)$ ((5.10)), and ν_n converges to ν_∞ in the W_1 -distance, by using the duality (5.12), we have that

$$\begin{aligned} &\left| \mathbb{E}^{\nu_n}[f_1(B_{t_1}^n) \cdots f_k(B_{t_k}^n)] - \mathbb{E}^{\nu_\infty}[f_1(B_{t_1}^\infty) \cdots f_k(B_{t_k}^\infty)] \right| \\ &= \left| \int_X \mathcal{P}_k^n d\nu_n - \int_X \mathcal{P}_k^\infty d\nu_\infty \right| \\ &\leq \left| \int_X \mathcal{P}_k^n d\nu_n - \int_X \widetilde{\mathcal{P}}_k^n d\nu_\infty \right| + \left| \int_X \widetilde{\mathcal{P}}_k^n d\nu_\infty - \int_X \mathcal{P}_k^\infty d\nu_\infty \right| \\ &\leq L W_1(\nu_n, \nu_\infty) + \|\widetilde{\mathcal{P}}_k^n - \mathcal{P}_k^\infty\|_{\infty, X_\infty} \int_X d\nu_\infty \\ &\rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Thus we have completed the proof. \square

We now show the weak convergence of finite-dimensional distributions for the case that initial distributions are $\frac{1}{m_n(X_n)}m_n$, which corresponds to the case of $m_n(X_n) < \infty$.

Lemma 5.5 *Let $m_n(X_n) < \infty$ for any $n \in \mathbb{N}$. Then, for any $k \in \mathbb{N}$, $0 = t_0 < t_1 < t_2 < \dots < t_k < \infty$ and $f_1, f_2, \dots, f_k \in C_b(X)$, the following holds:*

$$\mathbb{E}^{\tilde{m}_n}[f_1(B_{t_1}^n) \cdots f_k(B_{t_k}^n)] \xrightarrow{n \rightarrow \infty} \mathbb{E}^{\tilde{m}_\infty}[f_1(B_{t_1}^\infty) \cdots f_k(B_{t_k}^\infty)]. \quad (5.13)$$

Proof. Because of $m_n(X_n) < \infty$, we have $f \in L^2(X, m_n)$ for all $f \in C_b(X)$ for any $n \in \mathbb{N}$. Since \tilde{m}_n converges weakly to \tilde{m}_∞ in $\mathcal{P}(X)$, for any $\varepsilon > 0$, there exists a compact set $K \subset X$ so that

$$\sup_{n \in \mathbb{N}} \tilde{m}_n(K^c) < \varepsilon.$$

Thus, by (5.6), for any $\delta > 0$, there exists a compact set $K \subset X$ so that

$$\sup_{n \in \mathbb{N}} \left| \int_{X_n} \mathcal{P}_k^n d\tilde{m}_n - \int_K \mathcal{P}_k^\infty d\tilde{m}_\infty \right| \leq \left(\prod_{i=1}^k \|f_i\|_\infty \right) \sup_{n \in \mathbb{N}} \tilde{m}_n(K^c) < \delta. \quad (5.14)$$

Take $r > 0$ so that $K \subset B_r(\bar{x}_n) := \{x \in X : d(\bar{x}_n, x) < r\}$. Let $\tilde{\mathbf{1}}_r^R$ denote the following function: ($r < R$)

$$\tilde{\mathbf{1}}_r^R(x) = \begin{cases} 1 & x \in B_r(\bar{x}_n), \\ 1 - \frac{d(x, B_r(\bar{x}_n))}{R - r} & x \in B_R(\bar{x}_n) \setminus B_r(\bar{x}_n), \\ 0 & \text{o.w.} \end{cases}$$

Then $\tilde{\mathbf{1}}_r^R \in C_{bs}(X)$. Thus, by Theorem 2.12 and (5.14), for any $\delta > 0$, there exists $r > 0$ so that

$$\begin{aligned} & \left| \mathbb{E}^{\tilde{m}_n}[f_1(B_{t_1}^n) \cdots f_k(B_{t_k}^n)] - \mathbb{E}^{\tilde{m}_\infty}[f_1(B_{t_1}^\infty) \cdots f_k(B_{t_k}^\infty)] \right| \\ &= \left| \int_{X_n} \mathcal{P}_k^n d\tilde{m}_n - \int_{X_\infty} \mathcal{P}_k^\infty d\tilde{m}_\infty \right| \\ &= \left| \int_{X_n} \mathcal{P}_k^n d\tilde{m}_n - \int_{X_n} \tilde{\mathbf{1}}_r^R \mathcal{P}_k^n d\tilde{m}_n + \int_{X_n} \tilde{\mathbf{1}}_r^R \mathcal{P}_k^n d\tilde{m}_n - \int_{X_n} \tilde{\mathbf{1}}_r^R \mathcal{P}_k^\infty d\tilde{m}_\infty \right. \\ & \quad \left. + \int_{X_n} \tilde{\mathbf{1}}_r^R \mathcal{P}_k^\infty d\tilde{m}_\infty - \int_{X_\infty} \mathcal{P}_k^\infty d\tilde{m}_\infty \right| \\ &\leq \delta + \left| \int_{X_n} \tilde{\mathbf{1}}_r^R \mathcal{P}_k^n d\tilde{m}_n - \int_{X_n} \tilde{\mathbf{1}}_r^R \mathcal{P}_k^\infty d\tilde{m}_\infty \right| + \delta \\ &\xrightarrow{n \rightarrow \infty} 2\delta. \end{aligned}$$

Here, in the fifth line above, in the first δ , we used (5.14) and in the second δ , we used the tightness of the single measure m_∞ . The middle term in the fifth line converges to zero because of the L^2 -strong convergence of the heat semigroup P_t in the sense of Definition 2.11. Note that the total mass $m_n(X_n) \rightarrow m_\infty(X_\infty) \leq \infty$ because of the pmG convergence. Thus we have completed the proof. \square

Now we show the tightness of $\{\mathbb{B}^{\tilde{m}_n}\}$. For later arguments, we show the tightness for more general initial distributions ν_n than \tilde{m}_n .

Lemma 5.6 *Let $\nu_n \in \mathcal{P}(X_n)$ satisfy the following conditions:*

- (i) $\nu_n \rightarrow \nu_\infty$ weakly in $\mathcal{P}(X)$;

(ii) ν_n is absolutely continuous with respect to m_n with $d\nu_n = \phi_n dm_n$ and there exists a positive constant M so that, for any $r > 0$,

$$\sup_{n \in \mathbb{N}} \|\phi_n\|_{\infty, B_r(\bar{x}_n)} < M < \infty.$$

Then $\{\mathbb{B}^{\nu_n}\}_{n \in \mathbb{N}}$ is tight in $\mathcal{P}(C([0, \infty), X))$.

Proof.

Let us denote the law of $h(B^n)$ for $h \in \text{Lip}_b(X) \cap_{n=1}^{\infty} L^2(X; m_n)$ as follows:

$$\mathbb{B}^{\nu_n, h} = (h(B^n), \mathbb{P}_n^{\nu_n}).$$

It is easy to show that $\text{Lip}_b(X) \cap_{n=1}^{\infty} L^2(X; m_n)$ strongly separates points in $C_b(X)$, that is, for every x and $\varepsilon > 0$, there exists a finite set $\{h_i\}_{i=1}^l \subset \text{Lip}_b(X) \cap_{n=1}^{\infty} L^2(X; m_n)$ so that

$$\inf_{y: d(y, x) \geq \varepsilon} \max_{1 \leq i \leq l} |h_i(x) - h_i(y)| > 0.$$

Therefore, by [27, Corollary 3.9.2] with Lemma 5.4, the following two statements are equivalent:

- (i) $\{\mathbb{B}^{\nu_n}\}_{n \in \mathbb{N}}$ is tight in $\mathcal{P}(C([0, \infty), X))$;
- (ii) $\{\mathbb{B}^{\nu_n, h}\}_{n \in \mathbb{N}}$ is tight in $\mathcal{P}(C([0, \infty), \mathbb{R}))$.

Thus we will show that, for any $h \in \text{Lip}_b(X) \cap_{n=1}^{\infty} L^2(X; m_n)$,

$$\{\mathbb{B}^{\nu_n, h}\}_{n \in \mathbb{N}} \quad \text{is tight in} \quad \mathcal{P}(C([0, \infty); \mathbb{R})). \quad (5.15)$$

We note that, although [27, Corollary 3.9.2] gives sufficient conditions for tightness only in the càdlàg space $D([0, \infty); X)$, since the laws of each Brownian motion $\mathbb{B}_n^{\tilde{m}_n}$ have their support on the space of continuous paths $C([0, \infty); X)$, the tightness in $D([0, \infty); X)$ implies the tightness in $C([0, \infty), X)$. See, e.g., [28, Lemma 5 in Appendix] for this point.

Since ν_n converges weakly to ν_{∞} in $\mathcal{P}(X)$, the laws of the initial distributions $\{(h(B_0^n), \mathbb{P}_n^{\nu_n})\}_{n \in \mathbb{N}} = \{h_{\#} \tilde{m}_n\}_{n \in \mathbb{N}}$ is clearly tight in $\mathcal{P}(\mathbb{R})$. For $\delta > 0$, let us define

$$L_{\eta, T}^{n, h}(x) := \mathbb{P}_n^x \left(\sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq \eta}} |h(B_t) - h(B_s)| > \delta \right).$$

The desired result we would like to show is the following:

$$\lim_{\eta \rightarrow 0} \sup_{n \in \mathbb{N}} \int_{X_n} L_{\eta, T}^{n, h} \nu_n = 0, \quad (5.16)$$

for any $T > 0$. By the conditions (i) and (ii) in this lemma, for any $\varepsilon > 0$, there exists $R > 0$ so that

$$\begin{aligned} \int_{X_n} L_{\eta, T}^{n, h} \nu_n &= \|\phi_n \mathbf{1}_{B_R(\bar{x}_n)}\|_{\infty} \int_{X_n} L_{\eta, T}^{n, h} \mathbf{1}_{B_R(\bar{x}_n)} m_n + \nu_n(B_R^c(\bar{x}_n)) \\ &< M L_{\eta, T}^{n, h} \mathbf{1}_{B_R(\bar{x}_n)} m_n + \varepsilon. \end{aligned}$$

It suffices to show, for any $T, R > 0$,

$$\lim_{\eta \rightarrow 0} \sup_{n \in \mathbb{N}} \int_{X_n} L_{\eta, T}^{n, h} \mathbf{1}_{B_R(\bar{x}_n)} m_n = 0.$$

Let $m_{n, R} := \mathbf{1}_{Y_n} m_n$ whereby

$$Y_n = \overline{B_R(\bar{x}_n)}$$

is the closure of the open ball $B_r(\bar{x}_n)$. We have

$$\int_{X_n} L_{\eta, T}^{n, h} m_{n, R} = \mathbb{P}_{n, R+r}^{m_{n, R}} \left(\sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq h}} |h(B_t^n) - h(B_s^n)| > \delta : \Lambda_r \right) \quad (5.17)$$

$$+ \mathbb{P}_{n, R}^{m_{n, R}} \left(\sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq h}} |h(B_t^n) - h(B_s^n)| > \delta : \Lambda_r^c \right),$$

$$:= (\text{I})_{n, \eta} + (\text{II})_{n, \eta}. \quad (5.18)$$

whereby $\Lambda_r := \{w \in \Omega^n : \sup_{0 \leq t \leq T} d_n(B_t^n, B_0^n) < r\}$. Here $\mathbb{P}_{n, r}^x$ is a conservative diffusion process associated with $(\text{Ch}_n^r, \mathcal{F}_n^r)$, which is the Cheeger energy Ch on the closed ball Y :

$$\text{Ch}_n^r(f) = \frac{1}{2} \int_{Y_n} |\nabla f|_{w, Y_n}^2 m_{n, r}, \quad \mathcal{F}_n^r := \{f \in L^2(Y_n; m_{n, r}) : \text{Ch}_n^r(f) < \infty\}.$$

Recall that $|\nabla f|_{w, Y_n}^2$ means the minimal weak upper gradient on Y_n (see Subsection 2.4.2). We note that the Cheeger energy Ch_n^r on the closed ball Y_n is also quadratic because of [7, Theorem 4.19]. Since closed balls are not necessarily convex subset in X_n , the closed ball Y_n is not necessarily an $\text{RCD}(K, \infty)$ space. However, we can still construct the Brownian motion on Y_n since we have that $(\text{Ch}_n^r, \mathcal{F}_n^r)$ is quadratic ([7, Theorem 4.19]) and $[d(x, \cdot)] \leq m_{n, r}$ ([7, (iv) Theorem 4.18]) for any fixed $x \in Y_n$, which imply that $(\text{Ch}_n^r, \mathcal{F}_n^r)$ becomes a quasi-regular Dirichlet form by the same manner of [7, Lemma 6.7] and [10, Theorem 1.2] (see also [4, §7.2]). Here $[f]$ means the energy measure of the Cheeger energy (see [7, (4.21)]) and $[d(\bar{x}_n, \cdot)] \leq m_{n, r}$ means

$$\frac{d[d(\bar{x}_n, \cdot)]}{dm_{n, r}}(y) \leq 1 \quad m_{n, r}\text{-a.e. } y \in Y_n.$$

Note that although [7, Lemma 6.7] assumed the $\text{RCD}(K, \infty)$ condition, only the quadraticity of the Cheeger energy and $[d(\bar{x}_n, \cdot)] \leq m_{n, r}$ are used to construct the Brownian motions, and the $\text{CD}(K, \infty)$ condition is not necessary (see also [47, §4] for more detailed studies of the Cheeger energies and Brownian motions on subsets in $\text{RCD}(K, \infty)$ spaces).

We first estimate $(\text{I})_{n, \eta}$. By Lyons-Zheng decomposition ([49], and see also [30, Section 5.7]), we have

$$h(B_t^n) - h(B_s^n) = \frac{1}{2}(M_t^{[h]} - M_s^{[h]}) + \frac{1}{2}(M_{T-t}^{[h]}(r_T) - M_{T-s}^{[h]}(r_T)), \quad \mathbb{P}_{R+r}^{m_{n, R}}\text{-a.e.}, \quad (5.19)$$

for $0 \leq t \leq T$.

Then by time-symmetry (see [30, Lemma 5.7.1]), we have

$$\begin{aligned}
(I)_{n,\eta} &= \mathbb{P}_{R+r}^{m_{n,R}} \left(\sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq \eta}} |h(B_t^n) - h(B_s^n)| > \delta \right) \\
&\leq \mathbb{P}_{R+r}^{m_{n,R}} \left(\sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq \eta}} |M_t^{[h],n} - M_s^{[h],n}| > \delta \right) + \mathbb{P}_{R+r}^{m_{n,R}} \left(\sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq \eta}} |M_{T-t}^{[h],n}(r_T) - M_{T-s}^{[h],n}(r_T)| > \delta \right) \\
&= 2\mathbb{P}_{R+r}^{m_{n,R}} \left(\sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq \eta}} |M_t^{[h],n} - M_s^{[h],n}| > \delta \right).
\end{aligned} \tag{5.20}$$

Since $M^{[h],n}$ is a continuous martingale, by the martingale representation theorem, there exists the one-dimensional Brownian motion $\mathbf{B}^n(t)$ on an extended probability space $(\tilde{\Omega}, \tilde{\mathcal{M}}, \tilde{\mathbb{P}}_n^x)$ whereby $M^{[h],n}$ is represented as a time-changed Brownian motion with respect to the quadratic variation $\tilde{\mathbb{P}}_n^x$ -a.s, q.e. $x \in X_n$ (see, e.g., Ikeda–Watanabe [37, Chapter II Theorem 7.3]). That is, for q.e. $x \in X_n$,

$$M_t^{[h],n} = \mathbf{B}^n(\langle M^{[h],n} \rangle_t) = \mathbf{B}^n\left(\int_0^t \frac{d\mu_{\langle h \rangle}^n}{dm_n}(B_u^n) du\right) = \mathbf{B}^n\left(\int_0^t |\nabla h|_{w, Y_n}^2(B_u^n) du\right) \quad \tilde{\mathbb{P}}_n^x\text{-a.s.} \tag{5.21}$$

The last equality followed from [7, (iv) Theorem 4.18]. Since $|\nabla h|_{w, Y_n} \leq \text{Lip}(h)$, we have

$$\begin{aligned}
&\{\omega \in \tilde{\Omega} : \sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq \eta}} |M_t^{[h],n} - M_s^{[h],n}| > \delta\} \\
&= \{\omega \in \tilde{\Omega} : \sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq \eta}} \left| \mathbf{B}^n\left(\int_0^t |\nabla h|_{w, Y_n}^2(B_u^n) du\right) - \mathbf{B}^n\left(\int_0^s |\nabla h|_{w, Y_n}^2(B_u^n) du\right) \right| > \delta\} \\
&\subset \{\omega \in \tilde{\Omega} : \sup_{\substack{0 \leq s, t \leq \text{Lip}(h)^2 T \\ |t-s| \leq \text{Lip}(h)^2 \eta}} |\mathbf{B}^n(t) - \mathbf{B}^n(s)| > \delta\}.
\end{aligned}$$

Let \mathbb{W} be the standard Wiener measure on $C([0, \infty); \mathbb{R})$. Let

$$\theta(\eta, h) := \mathbb{W}_n\left(\sup_{\substack{0 \leq s, t \leq \text{Lip}(h)^2 T \\ |t-s| \leq \text{Lip}(h)^2 \eta}} |\omega(t) - \omega(s)| > \delta\right).$$

By (5.20) and noting $\sup_{n \in \mathbb{N}} m_n(B_R(\bar{x}_n)) < \infty$ because of the weak convergence of m_n , we have, for any $T > 0$,

$$\begin{aligned}
(I)_{n,\eta} &= \sup_{n \in \mathbb{N}} \int_{X_n} L_{\eta, T}^{n, h} m_{n, R} \\
&\leq \sup_{n \in \mathbb{N}} 2\mathbb{P}^{m_{n,R}} \left(\sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq \eta}} |M_t^{[h]} - M_s^{[h]}| > \delta \right) \\
&\leq 2\theta(\eta, h) \sup_{n \in \mathbb{N}} m_n(B_R(\bar{x}_n)) \\
&\xrightarrow{\eta \rightarrow 0} 0.
\end{aligned} \tag{5.22}$$

We now estimate $(\text{II})_{n,\eta}$. We have the following estimate:

$$\begin{aligned}
(\text{II})_{n,\eta} &= \mathbb{P}^{m_{n,R}} \left(\sup_{\substack{0 \leq s, t \leq T \\ |t-s| \leq h}} |h(B_t^n) - h(B_s^n)| > \delta : \Lambda_r^c \right) \\
&\leq 6m_n(B_{R+r}(\bar{x}_n)) \frac{1}{\sqrt{2\pi}} \int_{\frac{2r}{3\sqrt{\text{Lip}(h)^2 T}}}^{\infty} \exp\left\{-\frac{s^2}{2}\right\} ds \\
&\leq c \exp\{c_2(R+r)^2\} \int_{\frac{2r}{3\sqrt{\text{Lip}(h)^2 T}}}^{\infty} \exp\left\{-\frac{s^2}{2}\right\} ds. \\
&\leq c \exp\{c_2(R+r)^2\} \frac{3\sqrt{\text{Lip}(h)^2 T}}{2r} \exp\left\{-\frac{r^2}{18\text{Lip}(h)^2 T}\right\} \\
&\xrightarrow{r \rightarrow \infty} 0.
\end{aligned} \tag{5.23}$$

Here $c > 0$ is a constant independent of n . In the second line above, we used [30, Lemma 5.7.2], in the third line, we used the volume growth estimate (1.2) and, in the fourth line, we used the fact $\int_x^\infty \exp\{\frac{s^2}{2}\} ds \leq \frac{1}{x} \exp\{-\frac{x^2}{2}\}$. Thus, by (5.22) and (5.23), we have that, for any $R > 0$,

$$\limsup_{\eta \rightarrow 0} \sup_{n \in \mathbb{N}} \int_{X_n} L_{\eta,T}^{n,h} \mathbf{1}_{B_R(\bar{x}_n)} m_n = \limsup_{\eta \rightarrow 0} \sup_{n \in \mathbb{N}} \left((\text{I})_{n,\eta} + (\text{II})_{n,\eta} \right) = 0.$$

Thus we have the desired result (5.16). \square

We resume to prove Theorem 1.2.

Proof of Theorem 1.2. It is easy to check that the conditions (i) and (ii) in Lemma 5.6 are satisfied with $\nu_n = \tilde{m}_n$ in the both cases of $m_n(X_n) = \infty$ and $m_n(X_n) < \infty$. Thus we have shown the tightness. By using Lemma 5.5, we have completed the proof of (i) \implies (ii) in Theorem 1.2 in the case of $m_n(X_n) < \infty$. Moreover, we can check easily that the conditions in Lemma 5.4 are satisfied with $\nu_n = \tilde{m}_n$ in the case of $m_n(X_n) = \infty$ (see [32, Remark 4.6]). Therefore, we have completed the proof of (i) \implies (ii) in Theorem 1.2 in the case of $m_n(X_n) = \infty$. We finish the proof of (i) \implies (ii) in Theorem 1.2. \square

6 Proof of Theorem 1.4

To show the statement $(\text{iii})_{>0}$, it suffices to show the following statement:

(iii) $_{\varepsilon}$ There exist a complete separable metric space (X, d) and isometric embeddings $\iota_n : X_n \rightarrow X$ ($n \in \overline{\mathbb{N}}$) so that, for any $\varepsilon > 0$, it holds that

$$(\iota_n(B^n), \mathbb{P}_n^{\bar{x}_n}) \rightarrow (\iota_\infty(B^\infty), \mathbb{P}_\infty^{\bar{x}_\infty}) \quad \text{weakly in } \mathcal{P}(C([\varepsilon, \infty); X)). \tag{6.1}$$

Hereafter, we show the statement $(\text{iii})_{\varepsilon}$.

Proof of Theorem 1.4. We first show the case of the condition **(A)**, that is, $m_n(X_n) < \infty$.

The case of the condition **(A)**

Since we have already shown the weak convergence of the finite-dimensional distributions under the general $\text{RCD}(K, \infty)$ condition for starting points \bar{x}_n in Lemma 5.1, it suffices to prove the tightness:

Lemma 6.1 $\{\mathbb{B}_n^{\bar{x}_n}\}_{n \in \mathbb{N}}$ is tight in $\mathcal{P}(C([\varepsilon, \infty), X))$ for any $\varepsilon > 0$.

Proof. In the proof of [32, Theorem 7.7], we have

$$\sup_{n \in \mathbb{N}} \text{Ent}_{m_n}(p_n(t, \bar{x}_n, dy)) = \sup_{n \in \mathbb{N}} \text{Ent}_{m_n}(\mu_\varepsilon^{n, \bar{x}_n}) < \infty. \quad (6.2)$$

Let $\mathbb{B}_n^{\bar{x}_n}$ and $\mathbb{B}_n^{\tilde{m}_n}$ be restricted to the path space $C([\varepsilon, \infty), X)$. By using Markov property, we have that

$$\frac{d\mathbb{B}_n^{\bar{x}_n}}{d\mathbb{B}_n^{\tilde{m}_n}} = p(\varepsilon, \bar{x}_n, B_\varepsilon^n).$$

In fact, we have that, for any Borel measurable functions $F : C([\varepsilon, \infty), X) \rightarrow \mathbb{R}$,

$$\begin{aligned} \mathbb{E}^{\bar{x}_n}(F(B_{\varepsilon+}^n)) &= \mathbb{E}^{\bar{x}_n}(\mathbb{E}^{B_\varepsilon^n}(F)) \\ &= \int_{X_n} \mathbb{E}^y(F(B^n)) p_n(t, \bar{x}_n, dy) \\ &= \int_{X_n} \mathbb{E}^y(F(B^n)) p_n(t, \bar{x}_n, y) \tilde{m}_n(dy) \\ &= \mathbb{E}^{\tilde{m}_n}(p_n(\varepsilon, \bar{x}_n, B_0^n) F(B^n)) \\ &= \mathbb{E}^{\tilde{m}_n}(p_n(\varepsilon, \bar{x}_n, B_\varepsilon^n) F(B_{\varepsilon+}^n)). \end{aligned}$$

Let us denote $\text{Ent}_\nu(\mu) = \text{Ent}(\mu|\nu)$. By the fact that $\sup_{n \in \mathbb{N}} \text{Ent}_{m_n}(p_n(t, \bar{x}_n, dy)) < \infty$ ([32, p. 58]), we have

$$\begin{aligned} \sup_{n \in \mathbb{N}} \text{Ent}(\mathbb{B}_n^{\bar{x}_n} | \mathbb{B}_n^{\tilde{m}_n}) &= \sup_{n \in \mathbb{N}} \int_{\Omega} p_n(\varepsilon, \bar{x}_n, B_\varepsilon^n) \log \left\{ p_n(\varepsilon, \bar{x}_n, B_\varepsilon^n) \right\} d\mathbb{P}_n^{\tilde{m}_n} \\ &= \sup_{n \in \mathbb{N}} \int_{X_n} P_\varepsilon^n \left(p_n(\varepsilon, \bar{x}_n, \cdot) \log \{ p_n(\varepsilon, \bar{x}_n, \cdot) \} \right) d\tilde{m}_n \\ &= \sup_{n \in \mathbb{N}} \int_{X_n} p_n(\varepsilon, \bar{x}_n, \cdot) \log \{ p_n(\varepsilon, \bar{x}_n, \cdot) \} d\tilde{m}_n \\ &= \sup_{n \in \mathbb{N}} \frac{1}{m_n(X_n)} \text{Ent}_{m_n}(p_n(\varepsilon, \bar{x}_n, dy)) < \infty. \end{aligned}$$

In the third line above, we used the invariance property of \tilde{m}_n with respect to the heat semigroup $\{P_t^n\}_{t \geq 0}$ whereby

$$\frac{1}{m_n(X_n)} \int_{X_n} P_t^n f dm_n = \frac{1}{m_n(X_n)} \int_{X_n} f dm_n.$$

Note that $\inf_{n \in \mathbb{N}} m_n(X_n) > 0$ because $m_\infty(X_\infty) > 0$ by assumption that m_∞ is non-zero, and $m_n(X_n) \rightarrow m_\infty(X_\infty)$. Since $\{\mathbb{B}_n^{\tilde{m}_n}\}_{n \in \mathbb{N}}$ is tight by Lemma 5.6, by using the tightness criterion with respect to the entropy [32, Proposition 4.1], we have the tightness of $\{\mathbb{B}_n^{\bar{x}_n}\}_{n \in \mathbb{N}}$. \square

Now we resume the proof of Theorem 1.4.

Proof of Theorem 1.4. By the weak convergence of the finite-dimensional distributions in Lemma 5.1, and the tightness in Lemma 6.1, we have finished the proof of Theorem 1.4 for the case **(A)**.

Now we show the case of the condition **(B)**.

The case of the condition **(B)**

By using Markov property, we have that, for any Borel measurable functions $F : C([\varepsilon, \infty), X) \rightarrow \mathbb{R}$,

$$\begin{aligned}\mathbb{E}^{\bar{x}_n}(F(B_{\varepsilon+}^n)) &= \mathbb{E}^{\bar{x}_n}\left(\mathbb{E}^{B_\varepsilon^n}(F(B^n))\right) \\ &= \int_{X_n} \mathbb{E}^y(F(B^n))p_n(t, \bar{x}_n, dy) \\ &= \mathbb{E}^{p_n(t, \bar{x}_n, dy)}(F(B^n)).\end{aligned}$$

By Theorem 2.6, it holds that $p_n(t, \bar{x}_n, dy) \rightarrow p_\infty(t, \bar{x}_\infty, dy)$ in W_2 -sense and thus also in W_1 -sense (see e.g., [73, Remark 6.6]). Therefore, the condition (i) in Lemma 5.6 holds with $\nu_n = p_n(t, \bar{x}_n, dy)$. Moreover, the condition (ii) with $\nu_n = p_n(t, \bar{x}_n, dy)$ in Lemma 5.6 also holds by the assumption **(B)**. Therefore, by Lemma 5.4, and Lemma 5.6 with $\nu_n = p_n(t, \bar{x}_n, dy)$, we have the desired result. We finished the proof of Theorem 1.4. \square

7 Proof of Theorem 1.7

Proof of Theorem 1.7:

The goal of the proof is to show that, for any $f \in C_{bs}(X)$ (recall $C_{bs}(X)$ means the set of bounded continuous functions with bounded supports), we have

$$\int_X f dm_n \rightarrow \int_X f dm_\infty \quad \text{as } n \rightarrow \infty. \quad (7.1)$$

We first consider the case of $K > 0$.

The case of $K > 0$:

Let λ_n^1 be the spectral gap of Ch_n :

$$\lambda_n^1 := \inf\left\{\frac{\text{Ch}_n(f)}{\|f\|_{L^2(m_n)}^2} : f \in \text{Lip}(X_n) \setminus \{0\}, \int_{X_n} f dm_n = 0\right\}. \quad (7.2)$$

The following is a well-known fact (easy to obtain by using the spectral resolution):

$$\|P_t^n - m_n(\cdot)\|_{2 \rightarrow 2} \leq e^{-\lambda_n^1 t} \quad \forall t > 0, \quad (7.3)$$

whereby $\|\cdot\|_{2 \rightarrow 2}$ means the operator norm from $L^2(X_n; m_n)$ to $L^2(X_n; m_n)$, and $m_n(f) := \frac{1}{m_n(X_n)} \int_{X_n} f dm_n$.

By (7.3) and the assumption (1.6), we have that, for any $t > t_*$ (t_* appeared in the assumption (1.6)),

$$\begin{aligned}\|p_n(t, \bar{x}_n, \cdot) - \frac{1}{m_n(X_n)}\|_{L^2(m_n)} &= \|(P_s^n - m_n(\cdot))p_n(t-s, \bar{x}_n, \cdot)\|_{L^2(m_n)} \\ &\leq e^{-\lambda_n^1 s} \|p_n(t-s, \bar{x}_n, \cdot)\|_{L^2(m_n)} \\ &= e^{-\lambda_n^1 s} (p_n(2(t-s), \bar{x}_n, \bar{x}_n))^{1/2} \quad (0 < s < t, \quad t_* < \varepsilon := t-s) \\ &< M^{1/2} e^{-\lambda_n^1 (t-\varepsilon)}.\end{aligned} \quad (7.4)$$

Since the global Poincaré inequality holds under the $\text{CD}(K, \infty)$ condition with a positive $K > 0$ (see e.g., [73, Theorem 30.25]), we have that there exists a positive constant $C_P = C_P(K)$ depending only on K so that

$$\inf_{n \in \mathbb{N}} \lambda_n > C_P > 0. \quad (7.5)$$

By the condition of $K > 0$, there exists a positive constant C so that $\sup_{n \in \mathbb{N}} m_n(X_n) < C$ (see [68, Theorem 4.26]). Thus, by the statement (iii) $_{>0}$, (7.4) and (7.5), we have that, for any $\delta > 0$

$$\begin{aligned} & \left| \int_X f dm_n - \int_X f dm_\infty \right| \\ &= \left| \int_X f dm_n - m_n(X_n) \mathbb{E}_n^{\bar{x}_n}(f(B_t^n)) + m_n(X_n) \mathbb{E}_n^{\bar{x}_n}(f(B_t^n)) - m_\infty(X_\infty) \mathbb{E}_\infty^{\bar{x}_\infty}(f(B_t^\infty)) \right. \\ & \quad \left. + m_\infty(X_\infty) \mathbb{E}_\infty^{\bar{x}_\infty}(f(B_t^\infty)) - \int_X f dm_\infty \right| \\ &\leq C \left(\int_X |p_n(t, \bar{x}_n, y) - \frac{1}{m_n(X_n)}| f dm_n + |\mathbb{E}_n^{\bar{x}_n}(f(B_t^n)) - \mathbb{E}_\infty^{\bar{x}_\infty}(f(B_t^\infty))| \right. \\ & \quad \left. + \int_X |p_\infty(t, \bar{x}_\infty, y) - \frac{1}{m_\infty(X_\infty)}| f dm_\infty \right) \\ &\leq C \left(\|f\|_{L^2(m_n)} \|p_n(t, \bar{x}_n, \cdot) - \frac{1}{m_n(X_n)}\|_{L^2(m_n)} + |\mathbb{E}_n^{\bar{x}_n}(f(B_t^n)) - \mathbb{E}_\infty^{\bar{x}_\infty}(f(B_t^\infty))| \right. \\ & \quad \left. + \|f\|_{L^2(m_\infty)} \|p_\infty(t, \bar{x}_\infty, \cdot) - \frac{1}{m_\infty(X_\infty)}\|_{L^2(m_\infty)} \right) \\ &\leq C \left(\|f\|_{L^2(m_n)} M e^{-\lambda_n^1(t-\varepsilon)} + |\mathbb{E}_n^{\bar{x}_n}(f(B_t^n)) - \mathbb{E}_\infty^{\bar{x}_\infty}(f(B_t^\infty))| + \|f\|_{L^2(m_\infty)} M e^{-\lambda_\infty^1(t-\varepsilon)} \right) \\ &\leq C \left(\|f\|_{L^2(m_n)} M e^{-C_P(t-\varepsilon)} + |\mathbb{E}_n^{\bar{x}_n}(f(B_t^n)) - \mathbb{E}_\infty^{\bar{x}_\infty}(f(B_t^\infty))| + \|f\|_{L^2(m_\infty)} M e^{-C_P(t-\varepsilon)} \right) \\ &\rightarrow \delta + 0 + \delta \quad \text{as } n \rightarrow \infty \text{ and sufficiently large } t. \end{aligned}$$

Thus we finish the proof of Theorem 1.7 for the case of $K > 0$.

The case of $\sup_{n \in \mathbb{N}} \text{diam}(X_n) < D$:

The case of $\sup_{n \in \mathbb{N}} \text{diam}(X_n) < D$ can be proved in the same way as the case of $K > 0$ since the local Poincaré inequality holds for any $\text{RCD}(K, \infty)$ spaces (see [60, Theorem 1.1]). If $\sup_{n \in \mathbb{N}} \text{diam}(X_n) < D$ holds, then the local Poincaré inequality means the global Poincaré inequality and the proof will be the same as the case of $K > 0$. Thus we finish the proof of Theorem 1.7. \square

8 Proof of Theorem 1.9

In this section, we prove Theorem 1.9. In the previous sections, we have already proved the implications (i) \iff (ii) and (iii) $_{\geq 0}$ (or (iii) $_{>0}$) \implies (i) (since under Assumption 1.8, the conditions assumed in Theorem 1.7 are satisfied (especially the condition (1.6) is satisfied by the uniform Gaussian upper heat kernel estimate (3.7)). Thus we only have to show the implication (i) \implies (iii) $_{\geq 0}$.

Proof of (i) \implies (iii) $_{\geq 0}$ in Theorem 1.9. Since we have already showed the weak convergence of the laws of finite-dimensional distributions in Lemma 5.1 for the general

RCD(K, ∞) case, what we should prove is only the tightness of the Brownian motions on $C([0, \infty]; X)$.

Lemma 8.1 $\{\mathbb{B}_n\}_{n \in \mathbb{N}}$ is tight in $\mathcal{P}(C([0, \infty), X))$.

Proof. Since x_n converges to x_∞ in (X, d) , the laws of the initial distributions $\{B_0^n\}_{n \in \mathbb{N}} = \{\delta_{x_n}\}_{n \in \mathbb{N}}$ is clearly tight in $\mathcal{P}(X)$. Thus it suffices to show the following (see [18, Theorem 12.3]): for each $T > 0$, there exist $\beta > 0$, $C > 0$ and $\theta > 1$ such that, for all $n \in \mathbb{N}$

$$\mathbb{E}^{x_n}[\tilde{d}^\beta(B_t^n, B_{t+h}^n)] \leq Ch^\theta, \quad (0 \leq t \leq T \quad \text{and} \quad 0 \leq h \leq 1), \quad (8.1)$$

whereby $\tilde{d}(x, y) := d(x, y) \wedge 1$. Take $\beta > 0$ such that $\beta/2 - \nu > 1$, and set $\theta = \beta/2 - \nu$ (recall that ν was given in (3.5)). By the Markov property, we have

$$\begin{aligned} & \text{L.H.S. of (8.1)} \\ &= \int_{X_n \times X_n} p_n(t, x_n, y) p_n(h, y, z) \tilde{d}^\beta(\iota_n(y), \iota_n(z)) m_n(dy) m_n(dz). \\ &\leq \int_{X_n \times X_n} p_n(t, x_n, y) p_n(h, y, z) d^\beta(\iota_n(y), \iota_n(z)) m_n(dy) m_n(dz). \end{aligned} \quad (8.2)$$

By the Gaussian heat kernel estimate (3.7), we have

$$\begin{aligned} & \int_{X_n} p_n(s, y, z) d^\beta(\iota_n(y), \iota_n(z)) m_n(dz) \\ &\leq \frac{C_1}{cs^\nu} \int_{X_n} \exp\left(-C_2 \frac{d_n(y, z)^2}{s}\right) d^\beta(\iota_n(y), \iota_n(z)) m_n(dz) \\ &\leq \frac{C_1}{cs^\nu} \int_{X_n} \exp\left(-C_2 \frac{d_n(y, z)^2}{s}\right) d_n^\beta(y, z) m_n(dz) \\ &\leq C_1 c^{-1} C_2^{2/\beta} C_3 s^{\beta/2-\nu} m_n(X_n) \sup_{y, z \in X_n} \left\{ \left(C_2 \frac{d_n(y, z)^2}{s} \right)^{\beta/2} \exp\left(-C_2 \frac{d_n(y, z)^2}{s}\right) \right\} \\ &\leq C_1 c^{-1} C_2^{2/\beta} M_\beta s^{\beta/2-\nu} \\ &= C_4 s^{\beta/2-\nu}, \end{aligned} \quad (8.3)$$

whereby $M_\beta := \sup_{t \geq 0} t^{\beta/2} \exp(-t)$, $C_3 = \sup_{n \in \mathbb{N}} m_n(X_n)$ and $C_4 = C_4(N, K, D, \beta) = C_1 c^{-1} C_2^{2/\beta} C_3 M_\beta$ is a constant dependent only on N, K, D, β (independent of n). Note that, since m_n converges weakly to m_∞ and $m_\infty(X_\infty) < \infty$ because of $\text{diam}(X_\infty) < D$, we have that $\sup_{n \in \mathbb{N}} m_n(X_n) = C_3 < \infty$.

By (8.3), we have

$$\begin{aligned} \text{R.H.S. of (8.2)} &\leq C_4 h^{\beta/2-\nu} \int_{X_n} p_n(t, x_n, y) m_n(dy) \\ &\leq C_4 h^{\beta/2-\nu}. \end{aligned} \quad (8.4)$$

Thus we finish the proof. \square

Thus we have completed the proof Theorem 1.9. \square

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